Department of Physics and Astronomy
University of Heidelberg

Master Thesis in Physics
submitted by Frank Rösler

Frank Rösler

born in Schwäbisch Hall (Germany)

2014

Surface operators in Chern-Simons theory

This Master Thesis has been carried out by Frank Rösler at the
Institute for Theoretical Physics in Heidelberg
under the supervision of
Jun.Prof. Daniel Roggenkamp

Abstract:

This thesis is concerned with the relationship of Chern-Simons gauge theory to two-dimensional conformal field theory. The Chern-Simons action is introduced and several concepts such as Line and Surface Operators are introduced. The interpretation of surface operators as boundary conditions is explained and a classification of these operators in the abelian theory due to Kapustin and Saulina is reviewed [1].

An attempt to understand surface operators in the non-abelian theory is made. It is shown that in $SU(2)$ Chern-Simons theory surface operators can only exist if the level of the theory is an even integer. This confirms a well-known result in conformal field theory known as ADE classification [2]. In the case of $SU(3)$ Chenr-Simons theory it is shown that no level dependence arises even for nontrivial surface operators.

Finally, several concepts of conformal field theory are reviewed and the fusion of surface operators is described in two-dimensional terms resulting in a composition rule for rational $\hat{\mathfrak{u}}(1)$ CFTs.

In the appendix, several basic principles are reviewed.

Contents

1. Introduction

Chern-Simons gauge theory is a three dimensional topological quantum field theory which has gained considerable interest by mathematical physicists in recent years. There are several reasons for this. First of all, Chern-Simons theory is a particularly simple example of a topological quantum field theory, therefore providing a nice playground on which many new ideas and methods can be tried out.

Secondly, there is an intimate connection to two-dimensional conformal field theory first discovered by Witten in 1988 [3] (see also [4, 5]). This has raised interest in Chern-Simons theory since it provides a method to study various aspects of 2d conformal field theories in 3d terms.

Interesting observables in Chern-Simons theory are line operators and surface operators which are topological defects localized on one-dimensional lines and surfaces, respectively, as well as local operators inserted on the junction of two line operators. The collection of surface- line- and local point operators turns out to have the structure of a monoidal 2-category, the monoidal structure being given by fusion of two operators. The fusion product has been constructed explicitly in Chern-Simons theory by Kapustin and Saulina in [1] and [6].

A surface operator supported on a surface Σ in a theory TFT can (locally) be equivalently viewed as a boundary condition in the theory $TFT \otimes \overline{TFT}$ (where $\overline{TFT}$ denotes the parity reverse of TFT) by folding the space-time manifold across Σ. This correspondence makes it interesting to study boundary conditions in topological quantum field theories defined on three dimensional manifolds with boundaries. For abelian Chern-Simons theory this has been done in [1] where the 2-category structure has been used to constrain the values of the Chern-Simons level at which certain surface operators can exist. More precisely, they investigated line operators puncturing a surface operator leading to certain relations between their charges which are only consistent for certain values of the Chern-Simons level.

The present thesis is a first attempt on generalizing the results of [1] to the case of non-abelian Chern-Simons theory. The $SU(2)$ case turns out to be particularly interesting because of a known relationship to two-dimensional conformal field theory. It was shown e.g. in [4, 5] that boundary conditions in Chern-Simons theory lead to certain conformal field theories on the boundary (see also [7]). Several aspects of this relationship are outlined in sections 2.3 and 5.6.

In particular, it is known that a boundary condition (or equivalently, a surface operator) in $SU(2)$ Chern-Simons theory at level k leads to a CFT with symmetry $\hat{\mathfrak{su}}(2)_k$. But for such CFTs there exists an ADE classification [2]. The interpretation of surface operators as boundary conditions suggests that an analogous ADE classification should exist for surface operators in the three dimensional theory. We construct the surface operators corresponding to the A and D invariants by topological considerations. Especially, we will show that the surface operators corresponding to the D invariants can only exist for even k which is expected from analogous results in comformal field theory. We will also explicitly show an analogous result for surface operators in $SU(3)$ Chern-Simons theory, thereby confirming again well-known results in $SU(3)$ conformal field theories [22].

The E invariants in $\hat{\mathfrak{su}}(2)_k$ conformal field theories only exist for three special values of the level k and it does not seem likely that one can understand them by classical considerations. We will leave them for future work.

As a second step towards understanding the relationship between Chern-Simons theory and two-dimensional conformal field theory we will study the fusion of surface operators in the abelian theory from the two-dimensional point of view. To every surface operator in $U(1)$ Chern-simons theory there corresponds a free Boson-CFT on its suppport surface. The fusion of two surface operators will lead to a (kind of) product structure which takes two CFTs and assigns to them a new one.

In sections 2 to 4 we will mostly review known facts and recent developments in Chern-Simons theory. The following sections contain treatments of new, previously unsolved questions and comparisons to previously known results. In sections 2 and 3 we will first introduce the Chern-Simons action, as well as line and surface operators in Chern-Simons theory. In section 4, we will review some results of [1] which we need to compare with our own results in sections 6,7 and 5.6 where we study non-abelian Chern-Simons theory and two-dimensional conformal field theory, respectively. The appendix contains a more elementary and mathematical introduction to Chern-Simons theory.

2. Chern-Simons theory

2.1. The action

Chern-Simons gauge theory with gauge group G is defined by the action

$$S = \frac{k}{8\pi^2} \int_M \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big) \qquad (\mathrm{mod}\ 1) \tag{2.1.1}$$

for some closed three-manifold M and G-connection A. Tr denotes some Ad-invariant bilinear form on the Lie algebra $\mathfrak{g}$. The integer k is called the *level* of the theory (see appendix A for more details).

For non-simply connected gauge groups this action is not always well defined since there might not be a global section of the G-bundle to pull the connection back to M. One method to resolve this ambiguity is to choose a 4-manifold B such that $\partial B = M$ and define

$$S = \frac{k}{8\pi^2} \int_B \mathrm{Tr}(\tilde{F} \wedge \tilde{F}) \qquad (\mathrm{mod}\ 1), \tag{2.1.2}$$

where $\tilde{F}$ extends the curvature F over B. Since $d(F \wedge F) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$, this definition coincides with (2.1.1) when (2.1.1) makes sense, but also exists in more general cases, namely whenever the bundle $E \to M$ extends to a bundle over B. As pointed out in [8], for an abelian gauge group this is always possible. The case of even more general compact gauge groups requires special treatment. A good reference for this is [23].

For the action to be independent of the bounding manifold B one must require that for *closed* 4-manifolds B, the integral (2.1.2) is an integer. This fixes the normalization of the bilinear form Tr.

2.2. The abelian case

Let now $G = U(1)$. Then $\mathfrak{g} = i\mathbb{R}$ and

$$S = \frac{k}{4\pi^2} \int_B \tilde{F} \wedge \tilde{F}, \tag{2.2.1}$$

(where the bilinear form Tr is encoded in $\frac{k}{4\pi^2}$). In this normalization the path integral reads

$$Z(M) = \int \exp\left(2\pi i \frac{k}{4\pi^2}\int_B \tilde{F}\wedge\tilde{F}\right)\mathcal{D}A. \tag{2.2.2}$$

It is convenient to redefine the action such that the path integral reads

$$Z(M) = \int \exp\left(-S'(A)\right)\mathcal{D}A \tag{2.2.3}$$

which is achieved by putting

$$S'(A) = \frac{iK}{4\pi}\int_B \tilde{F}\wedge\tilde{F}. \tag{2.2.4}$$

where $K = 2k$ (note that now the action is defined modulo $2\pi i$ rather than modulo 1). This is the normalization used in [1].

For an arbitrary abelian gauge group $G = U(1)^N$ the even integer K is replaced by a symmetric even integral bilinear form on $\mathfrak{u}(1)^N$ i.e. a bilinear form with even integers on the diagonal which is contracted with the Lie algebra valued four-form $\tilde{F}\wedge\tilde{F}$.

When talking about the abelian theory, we will always use the action (2.2.4) and omit the prime.

In the abelian theory we have the big advantage that the Lie algebra of $G = U(1)^N$ is (very nearly) the same as the universal cover of G. We can think of G as the quotient $\mathbb{R}^n/2\pi\Lambda$ for a subgroup $\Lambda \cong \mathbb{Z}^n$ and of the Lie algebra $\mathfrak{g}$ as $\Lambda\otimes\mathbb{R}$. The integrality conditions on the bilinear form K now read $K(\lambda,\lambda')\in\mathbb{Z}$ and $K(\lambda,\lambda)\in 2\mathbb{Z}$ for $\lambda,\lambda'\in\Lambda$. This will be helpful in section 4.

2.3. From Chern-Simons theory to conformal field theory

As mentioned in the introduction, there is an interesting relationship between Chern-Simons theory and two-dimensional conformal field theory. We will explain the origin of this relationship following [4]; see also [3],[5].

We start with the action (2.1.1) on $M = \Sigma\times\mathbb{R}$ for some compact surface Σ. Decompose the exterior derivative as $d = dt\frac{\partial}{\partial t} + \tilde{d}$ where t is the coordinate in $\mathbb{R}$ and $\tilde{d}$ is the exterior derivative on Σ. Similarly, we write $A = A_0 + \tilde{A}$ for the gauge field. The action can be written as

$$S = -\frac{k}{8\pi^2}\int_M \mathrm{Tr}\left(\tilde{A}\wedge\frac{\partial\tilde{A}}{\partial t}dt\right) + \frac{k}{4\pi^2}\int_M \mathrm{Tr}\big(A_0\wedge(\tilde{d}\tilde{A}+\tilde{A}^2)\big). \tag{2.3.1}$$

(For the action to take this form, it is necessary to choose a boundary condition to kill surface terms.)

Obviously, A_0 can be integrated out in the path integral giving us a delta functional $\delta(\tilde{F})$, where $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}^2$.

As an example, let us implement this constraint in the case $\Sigma = D$ a disk. The constraint $\tilde{F} = 0$ is solved by $A = -\tilde{d}U\,U^{-1}$ for a map $U : \Sigma \to G$. Plugging this back into the action, we obtain

$$S = \frac{k}{8\pi^2}\int_{\partial M} \mathrm{Tr}\big(U^{-1}\partial_\phi U\,U^{-1}\partial t U\big)\,d\phi\,dt + \frac{k}{24\pi^2}\int_M \mathrm{Tr}\big((U^{-1}dU)^3\big), \tag{2.3.2}$$

where ϕ is the angular coordinate on ∂D.

This is the action of a two-dimensional conformal field theory known as the Wess-Zumino-Witten (WZW) model.[1] It only depends on boundary values of the field U.

Note that different choices of boundary conditions yield different 2d CFTs, since the boundary values of the gauge transformations U depend on the boundary condition.

The connection between Chern-Simons theory and 2d WZW models can also be seen in different ways. In [3], for example, it was noted that the Hilbert space of states of Chern-Simons theory can be identified with the space of conformal blocks of the WZW model. In [9] an isomorphism between the classical phase spaces of the Chern-Simons and WZW theories is constructed. Yet another way to see the connection is presented in [10].

In section 5.6 we will explore this relationship in the case of $U(1)$ Chern-Simons theory with surface operators and line operators inserted. It is well-known that line operators puncturing a surface operator correspond to primary fields in the corresponding CFT [6]. In section 5 we will show this relation ship in the case of $U(1)$ Chern-Simons theory explicitly.

[1] We note that one not only recovers the *action* of the WZW model from Chern-Simons theory, but even the WZW *path integral* since the change of variables from A to U has trivial Jacobian: $\int \delta(\tilde{F})\mathcal{D}\tilde{A} = \int \mathcal{D}U$.

3. Line and Surface Operators

3.1. Line Operators and the Discriminant Group

Line Operators are observables localized on one-dimensional curves and are an important tool to study the structure of topological and conformal field theories. In the following, we will give a brief characterization of line operators in abelian Chern-Simons theory which eventually leads to the notion of the *discriminant group*. We will mainly follow [1].

In abelian Chern-Simons theory, line operators are *Wilson lines*. Given a curve γ, a Wilson line is defined as

$$W_X(\gamma) = \exp\left(\int_\gamma X(A)\right), \tag{3.1.1}$$

where $X \in \Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$ is its *charge*. It is easy to see that for closed γ Wilson loops are gauge invariant: if f is a gauge transformation, A transforms as $A \mapsto A + df$ and the Wilson loop becomes

$$\begin{aligned}\exp\left(\int_\gamma X(A+df)\right) &= \exp\left(\int_\gamma X(A) + \int_\gamma X(df)\right)\\ &= \exp\left(\int_\gamma X(A) + \int_\gamma dX(f)\right)\\ &= \exp\left(\int_\gamma X(A)\right)\end{aligned}$$

by Stokes' theorem.

If the curve γ is not closed, Wilson lines may still exist if their charge can be screened by a local operator. Also, two Wilson lines with different charges may be connected by inserting a local operator at their joining point. In this way, one obtains the structure of a category whose objects are the line operators and whose morphisms are local operators inserted at their junction.

Let us go a little deeper into this. First, it is clear that the only local operators which can be inserted into closed Wilson loops are multiples of the identity (otherwise we would lose gauge invariance). In categorical language this means that the space of endomorphisms of a Wilson loop W_X is $\mathbb{C}$.

Now consider two Wilson lines W_X, W'_X joining at some point $p \in M$. Applying a gauge transformation f we get a contribution

$$\exp\big((X - X')(f(p))\big) \tag{3.1.2}$$

which has to be cancelled. It turns out that the only way to do this is using Dirac monopole operators [1]. These have electric charges Km, $m \in \Lambda$ (see section 3.1 of [1]).[1] As a result, two Wilson lines can be joined if and only if their charges satisfy

$$X - X' = Km \tag{3.1.3}$$

for some $m \in \Lambda$. It is clear that if there exists a morphism from W_X to W'_X, then there also exists one from W'_X to W_X (simply by changing the sign of m). So, regarded as category objects W_X and W_{X+Km} are isomorphic.

We can pass from the above category to an equivalent one by identifying isomorphic objects. The isomorphism classes of Wilson lines are now classified by the finite group

$$D = {}^{\Lambda^*}\big/_{\mathrm{im}(K)} \tag{3.1.4}$$

known as the *Discriminant group.*

3.1.1. Example

Let us compute the Discriminant group for an interesting special case. Consider the theory of section 2.2 with $G = U(1)$, $\Lambda = \Lambda^* = \mathbb{Z}$ and $K = 2k$.

Clearly, $\mathrm{im}(K) = 2k\mathbb{Z}$ and so

$$D = {}^{\Lambda^*}\big/_{\mathrm{im}(K)} = {}^{\mathbb{Z}}\big/_{2k\mathbb{Z}} \tag{3.1.5}$$

From this very simple calculation we immediately know that in $U(1)$ Chern-Simons theory with $K = 2k$ there are exactly $2k$ non-isomorphic line operators.

3.2. Surface Operators

Surface Operators in 3d topological quantum field theory are topological defects supported on a codimension one submanifold of M, i.e. they are defined by gluing together two different TQFTs defined on the two sides of the surface. They are important observables in 3d TFT and have gained much attention in recent years [11]-[12],[7],[13].

[1] Here, we are regarding the bilinear form K as a map $K : \Lambda \to \Lambda^*$.

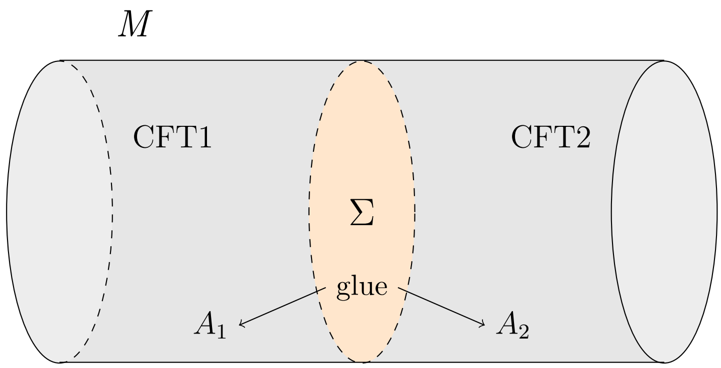


Figure 3.1.: A surface operator in a solid cylinder M.

There are several operations one can perform with surface operators and line operators which result in a nice mathematical structure. First, two surface operators S and S' may meet on a one-dimensional curve if a line operator W is inserted on their junction (equivalently, we can view the line operator as changing S to S').

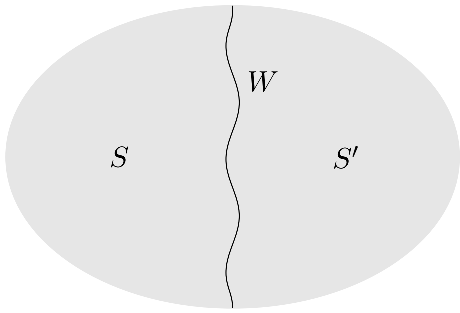


Figure 3.2.: A line operator W separating two surface operators S and S'.

This operation gives us the structure of a category whose objects are the surface operators itself and whose morphisms are line operators inserted between them. Similarly, line operators may be joined by inserting a local operator φ at their junction.

Thus, line operators form a category, too with morphisms given by local operators inserted at their junction. Altogether we obtain the structure of a 2-category (that is, a category whose morphisms are a category again).

There a notion of tensor produce for surface operators known as *fusion*. To get a feeling what this means, imagine we have two surface operators supported on two surfaces of the same topology. What will happen if we let the distance between the two surfaces go to zero?

The surfaces will merge and the theory between them will vanish resulting in a gluing

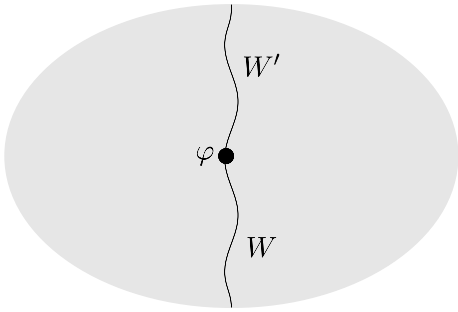


Figure 3.3.: A local operator φ separating two line operators W and W'.

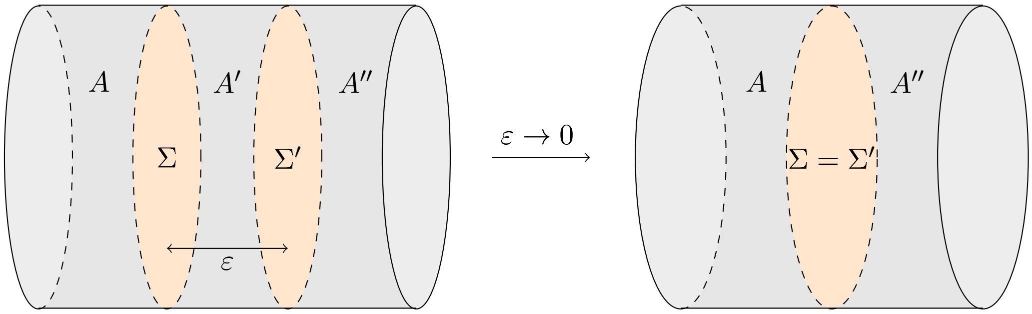


Figure 3.4.: The fusion of surface operators.

of the outer theories, i.e. a new surface operator. The fusion of surface operators equips our 2-category with a monoidal structure (that is, a sort of tensor product). More on this topic can be found in [11]. Explicit constructions of this monoidal structure have been given in abelian Chern-Simons theory by Kapustin and Saulina in [6] and [1]. We will come back to this topic in section 5.6.

We will end this section with a brief description of the so-called *folding trick*. Locally, a surface operator in a given CFT can be equivalently viewed as a boundary condition the tensor product $TFT \otimes \overline{TFT}$ of the theory with its parity reverse on the manifold with boundary that arises when one folds the space-time manifold across the surface where the defect is localized (see e.g. [14, 6]). More precisely, suppose we have a surface operator gluing the G-connections $A = A_\mu dx^\mu$ and $A' = A'_\mu dx^\mu$ on a surface Σ, i.e. imposing some relation between A and A' which we will call $\mathcal{R}(A, A')$.

We let the x^1-direction run perpendicular to Σ. Now, take the part of the space-time manifold lying on the right of Σ, reflect it across Σ and identify points with the part on the left. Because of the reflection the 1-component of A' picks up a sign so that we are now dealing with $\bar{A}' := -A'_1 dx^1 + A'_2 dx^2 + A'_3 dx^3$.

If we now take the theory with gauge group $G \oplus G$ whose gauge field is the pair $A, \bar{A}'$,

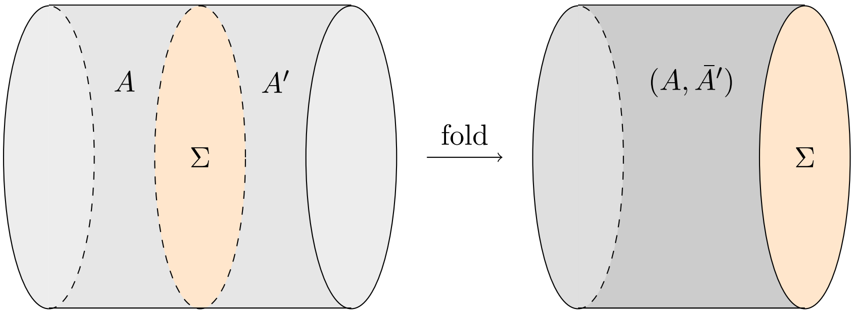


Figure 3.5.: .

it is clear that we have not lost any information compared to the unfolded theory. We are simply looking at it from a different perspective. But now remember that still the relation $\mathcal{R}(A, A')$ is imposed on Σ. This is nothing but a boundary condition! Thus, surface operators in some theory are locally equivalent to boundary conditions in the folded theory.

Using this trick, one can study certain properties of surface operators in TFTs by studying boundary conditions in different TFTs.

In special cases, surface operators can be implemented using a Lagrange multiplier field living on a surface. This is particularly convenient since many features of the operators can be seen from the classical action. We will pursue this method in section 4.2.

4. Topological boundary conditions in $U(1)^N$ Chern-Simons theory

In the following sections, we will study surface operators in Chern-Simons theory and various aspects of their relationship to two-dimensional conformal field theory. Many aspects of this relationship can be studied classically using only the action of the theory. Particularly in the abelian theory (which is a free theory) we expect the classical theory to be a good approximation. In [1], Kapustin and Saulina studied topological boundary conditions in abelian Chern-Simons theory (which, as we have seen, are equivalent to surface operators). We will give a short review of their methods and results.

4.1. The boundary term

Let us consider Chern-Simons theory with gauge group $G = U(1)^N$ on a 3-manifold M with boundary Σ. That is, we consider the action

$$S = \frac{i}{4\pi} \int_M K(A, dA) \qquad (\text{mod } 2\pi i) \tag{4.1.1}$$

where K is the even symmetric bilinear form mentioned in section 2.2. This theory is particularly nice since it is a *free* theory in which many features can be seen classically. Let us compute the Euler-Lagrange equations for this action. The variation of the action reads

$$\begin{aligned} \delta S &= \frac{i}{4\pi} \int_M K(\delta A, dA) + \frac{i}{4\pi} \int_M K(A, d\delta A) && (4.1.2) \\ &= \frac{i}{2\pi} \int_M K(\delta A, dA) - \int_\Sigma K(A, \delta A) && (4.1.3) \end{aligned}$$

The first term gives the familiar equation of motion $K^t(dA) = 0$, whereas the second term should not be there. We want the theory to be local which means that boundary values of A should not affect the equation of motion in the bulk. Thus, we must require this term to vanish which means that $A|_\Sigma$ should lie in a Lagrangian subspace of the Lie algebra $\mathfrak{g} = \Lambda \otimes \mathbb{R}$ with respect to the bilinear form K.

Due to the restriction of the gauge field to a Lagrangian subspace $L \subset \mathfrak{g}$ the gauge group will be reduced to a group smaller than $U(1)^N$ which has Lie algebra L. The identity component of this smaller group will be a torus which we think of as $L/\Lambda_0 =: T_{\Lambda_0}$ for an appropriate sublattice $\Lambda_0 \subset \Lambda$.

We will denote by P the inclusion of Λ_0 into Λ.

4.2. Imposing the condition

It turns out to be convenient and enlightening to use Lagrange multipliers to impose the boundary condition. Suppose we have chosen a certain Lagrangian sublattice $\Lambda_0 \subset \Lambda$. We choose a torus $T_\Phi = (\Phi \otimes \mathbb{R})/2\pi\Phi$, where Φ is a finitely-generated free abelian group whose rank is half the rank of Λ and a field

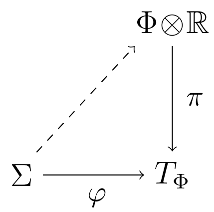
$$\varphi : \partial M \to T_\Phi \tag{4.2.1}$$

and add a term

$$\frac{i}{2\pi} \int_{\partial M} V(\varphi, dA) \tag{4.2.2}$$

to S. φ in (4.2.2) is understood to be lifted to $\Phi \otimes \mathbb{R}$ and V is a bilinear mapping $(\Phi \otimes \mathbb{R}) \times (\Lambda \otimes \mathbb{R}) \to \mathbb{R}$. Several comments are in order.

- First, it does not seem natural to choose a torus-valued field as Lagrange multiplier. However, we will soon see that this is necessary for gauge invariance.
- Second, we note that for general Σ there might not be a continuous lift of the field φ. To see this, consider the diagram



  Since the projection π happens to be a covering map, we can employ the universal lifting theorem which tells us that there exists a continuous lift if and only if $\varphi_* \pi_1(\Sigma) \subset \pi_* \pi_1(T_\Phi)$. This means $\varphi_* \pi_1(\Sigma) = \{1\}$.

  So, in general we cannot expect to obtain a *continuous* lift but have to accept discontinuities of φ around non-trivial loops in Σ.

- Finally, for (4.2.2) to be well-defined modulo $2\pi i$, the matrix V must obey certain integrality conditions which can be seen as follows. At a discontinuity, the lifted field φ is allowed to jump by 2π times a lattice vector ω of Φ (otherwise the original field in T_Φ would not be smooth). The term (4.2.2) changes by
$$\frac{i}{2\pi}\int_{\partial M} V(2\pi\omega, dA) = i\,V\left(\omega\,, \int_{\partial M} dA\right)$$
For this to vanish modulo $2\pi i$, the restriction of V to $\Phi\times\Lambda$ must take integral values. Thus, the restricted map V has to be regarded as an element of $\mathrm{Hom}(\Phi,\Lambda^*)$.

For (4.2.2) to give the right boundary condition we must have
$$\ker V^t = \Lambda_0. \tag{4.2.3}$$
It is shown in [1] that the boundary gauge group is disconnected if and only if $\mathrm{im}\,V$ is a proper sublattice of $\ker P^t$. If d denotes the order of $\ker P^t/\mathrm{im}\,V$ then for every u in the boundary gauge group, u^d lies in T_{Λ_0} so the boundary gauge group consists of several Lagrangian tori in G.

To maintain gauge invariance we need to choose a transformation law $\varphi \mapsto \varphi + \delta\varphi$ as follows. Under a gauge transformation f such that $f|_{\partial M} \in T_{\Lambda_0}$ the bulk action varies by a boundary term
$$-\frac{i}{4\pi}\int_{\partial M} K(f, dA). \tag{4.2.4}$$
The term (4.2.2) varies by
$$\frac{i}{2\pi}\int_{\partial M} V(\delta\varphi, dA). \tag{4.2.5}$$
Choose a homomorphism $W : \Lambda_0 \to \Phi$ satisfying
$$K|_{\Lambda_0} = 2VW \tag{4.2.6}$$
and denote the induced homomorphism between the tori $W_T : T_{\Lambda_0} \to T_\Phi$. Having found such a map W, gauge invariance is restored by putting
$$\delta\varphi = W_T(f) \tag{4.2.7}$$

4.3. The boundary gauge group

The full boundary gauge group now can be obtained as follows. Since the matrix V^t : $\Lambda \to \Phi^*$ is integral it induces a homomorphism between the tori
$$V_T^t : G \to T_{\Phi^*} \tag{4.3.1}$$

(where T_{Φ^*} is the torus belonging to Φ^*).

Then the boundary gauge group belonging to V is the kernel of V_T^t.

Let us demonstrate this at a simple example. Consider a surface operator in $U(1) \times U(1)$ Chern-Simons theory and take $\Lambda = \mathbb{Z}^2$ and $\Lambda_0 = \{(n,n)|n \in \mathbb{Z}\}$. In this case, we have

$$\begin{aligned} P : \mathbb{Z} &\hookrightarrow \mathbb{Z}^2 \\ n &\mapsto (n,n). \end{aligned}$$

We also choose $\Phi = \mathbb{Z}$. With this data given, the surface operator will be fully determined by a choice of the matrix V in (4.2.2). The matrix V must be chosen such that $\ker(V^t) = \Lambda_0$ which is satisfied by

$$V(m) = (vm, -vm)$$

for some integer v dividing the level k. Let us compute $\ker(P^t)$ now to see when $\mathrm{im}(V)$ is a proper sublattice of it. We have for $\xi = (\xi_1, \xi_2) \in \Lambda^*$

$$\begin{aligned} \langle P^t(\xi), n \rangle &= \langle \xi, P(n) \rangle \\ &= \xi(n,n) \\ &= \xi_1 n + \xi_2 n \\ &= (\xi_1 + \xi_2) n \end{aligned}$$

So clearly, $\ker(P^t) = \{(\xi_1,\xi_2) \in \Lambda^* |\, \xi_1 = -\xi_2\} \cong \{(m,-m)|\, m \in \mathbb{Z}\}$. This shows that $\mathrm{im}(V)$ is a proper sublattice of $\ker(P^t)$ if and only if $v > 1$.

Now, to link this to $\ker(V_T^t)$, we have to show that $\ker(V_T^t)$ is disconnected if and only if $\mathrm{im}(V)$ is a proper sublattice of $\ker(P^t)$ (that is, if and only if $v > 1$). In face, we will see that the number of connected components of $\ker(V_T^t)$ is v.

The proof of this is a straightforward calculation. Let $(x,y) \in \Lambda \otimes \mathbb{R}$. We have $V_T^t[(x,y)] = [V_{\mathbb{R}}^t(x,y)]$ by definition, where $V_{\mathbb{R}}^t$ is the induced map on $\Lambda \otimes \mathbb{R}$. So we have $(x,y) \in \ker V_T^t$ if and only if

$$\begin{aligned} & [V_{\mathbb{R}}^t(x,y)] = 0 \in T_{\Phi^*} \\ \Leftrightarrow \quad & V_{\mathbb{R}}^t(x,y) \in \Phi^* \\ \Leftrightarrow \quad & \langle V_{\mathbb{R}}^t(x,y), \varphi \rangle \in \mathbb{Z}\ \forall \varphi \in \Phi \\ \Leftrightarrow \quad & v \cdot (x-y) \cdot \varphi \in \mathbb{Z}\ \forall \varphi \in \Phi \\ \Leftrightarrow \quad & v \cdot (x-y) \in \mathbb{Z} \\ \Leftrightarrow \quad & [v \cdot (x-y)] = 0\ \in \mathbb{R}/2\pi\mathbb{Z} \end{aligned}$$

The set of solutions to this constraint is

$$\{(e^{ix}, e^{iy}) \in U(1) \times U(1) \mid y = x + 2\pi\tfrac{\ell}{v}, \ell \in \mathbb{Z}_v\} \tag{4.3.2}$$

which is isomorphic to $U(1) \times \mathbb{Z}_v$. This makes it very clear that indeed the number of connected components of $\ker(V_T^t)$ is v.

We will encounter the same phenomenon again when talking about the non-abelian case.

This ends our review of [1].

5. Surface Operators in $U(1)$ Chern-Simons theory and Conformal Field Theory

5.1. Classical treatment

Kapustin and Saulina whose results were reviewed in the previous sections have explicitly shown many features of surface operators in topological field theories. However, they said little about the relationship to two-dimensional conformal field theory. Many aspects of this relationship can be seen very explicitly in abelian Chern-Simons theory. We will now start studying surface operators in $U(1)$ Chern-Simons theory from different perspectives and compare our results to those cited in the previous sections. So let M be a closed, oriented 3-manifold and $\Sigma \subset M$ a closed oriented surface which we take to be the support of a surface operator. We could now apply the folding trick and use the results on boundary conditions from section 4 to learn a lot about them. But to be more pedagogical, let us try a more direct approach here and make contact with the results in section 4 in the end.

The gluing condition which defines the surface operator can be implemented either "by hand" or using a Lagrange multiplier field. We will treat both methods beginning by the realisation by hand.

5.2. Gluing by hand

Let M be a closed oriented three-manifold and let $\Sigma \subset M$ be a closed oriented surface. We will denote the pieces of M lying on the left resp. on the right of Σ by M_1 resp. M_2 and consider independent Chern-Simons theories on these pieces (see figure 3.1). The action is[1]

$$S = k\frac{i}{2\pi}\int_{M_1} A_1 \wedge dA_1 + k\frac{i}{2\pi}\int_{M_2} A_2 \wedge dA_2 \qquad (\mathrm{mod}\ 2\pi i). \tag{5.2.1}$$

[1] In this normalization, k is an arbitrary integer.

The fields A_j are gauge fields whose curvatures represent integral cohomology classes. Under a gauge transformation $f_j : M_j \to \mathbb{R}/2\pi\mathbb{Z}$ we have $A_j \mapsto A_j + df_j$ and the action changes by

$$S \mapsto S + k\frac{i}{2\pi}\int_{M_1} df_1 \wedge dA_1 + k\frac{i}{2\pi}\int_{M_2} df_2 \wedge dA_2 \tag{5.2.2}$$

$$= S + k\frac{i}{2\pi}\int_{\Sigma} f_1\, dA_1 - k\frac{i}{2\pi}\int_{\Sigma} f_2\, dA_2 \tag{5.2.3}$$

$$= S + k\frac{i}{2\pi}\int_{\Sigma}(f_1\, dA_1 - f_2\, dA_2). \tag{5.2.4}$$

We will have to make this term vanish modulo $2\pi i$.

Let us now compute the equation of motion for A. The variation of the action is

$$\begin{aligned}\delta S &= k\frac{i}{2\pi}\int_{M_1}(\delta A_1 \wedge dA_1 + A_1 \wedge d\delta A_1) + ...\\ &= k\frac{i}{2\pi}\int_{M_1}\delta A_1 \wedge dA_1 - k\frac{i}{2\pi}\int_{\Sigma} A_1 \wedge \delta A_1 + k\frac{i}{2\pi}\int_{M_1}\delta A_1 \wedge dA_1 + ...\\ &= k\frac{i}{\pi}\int_{M_1}\delta A_1 \wedge dA_1 + k\frac{i}{\pi}\int_{M_2}\delta A_2 \wedge dA_2 + k\frac{i}{2\pi}\int_{\Sigma}(A_2 \wedge \delta A_2 - A_1 \wedge \delta A_1),\end{aligned}$$

where the ellipses stand for the analogous terms on M_2. The first two terms give the usual equation of motion $dA_j = 0$ but the third term is unwanted because the values of the gauge fields on a surface should not influence the equation of motion in the bulk. A natural way to solve this problem is to introduce the *gluing condition*

$$A_1\big|_{\Sigma} = A_2\big|_{\Sigma}. \tag{5.2.5}$$

Of course it should not be possible to violate this equation by applying gauge transformations so we also get the restriction

$$df_1 = df_2 \qquad \text{on } \Sigma. \tag{5.2.6}$$

This means that on Σ we do not have the full $U(1) \times U(1)$ gauge freedom which would be generated by f_1 and f_2 but only a subgroup of it.

Equation (5.2.6) implies $f_1 - f_2 = \text{const} =: r$ on Σ. We want the subgroup generated by these constrained gauge transformations to be compact which is guaranteed if and only if $r \in \mathbb{Q}$.

Now let us turn to gauge invariance again. We have seen in (5.2.4) that the term

$$k\frac{i}{2\pi}\int_{\Sigma}(f_1\, dA_1 - f_2\, dA_2) \tag{5.2.7}$$

should vanish modulo $2\pi i$. Now we can use our conditions (5.2.5) and (5.2.6) to enforce this. Let us put $f_1 - f_2 = 2\pi\frac{\ell}{b}$ with $\ell \in \mathbb{Z}_b$ (this suffices since $f_1 + 2\pi\frac{\ell+nb}{b} = f_1 + 2\pi\frac{\ell}{b} \pmod{2\pi}$). Remembering that we have $A_1 = A_2$ on Σ we get

$$\begin{aligned} k\frac{i}{2\pi}\int_\Sigma (f_1\, dA_1 - f_2\, dA_2) &= k\frac{i}{2\pi}\int_\Sigma (f_1\, dA_1 - f_2\, dA_1) \\ &= k\frac{i}{2\pi}\int_\Sigma (f_1 - f_2) dA_1 \\ &= \frac{k\ell}{b} i \int_\Sigma dA_1 \end{aligned}$$

Since $\int_\Sigma dA_1 \in 2\pi\mathbb{Z}$ for closed surfaces Σ we obtain the constraint

$$\frac{k\ell}{b} \in \mathbb{Z} \tag{5.2.8}$$

which is equivalent to $b|k$.

Thus, we have found that the gauge groups which can be produced by surface operators are in correspondence with divisors b of the level k and are isomorphic to $U(1) \times \mathbb{Z}_b$.

5.3. Lagrange multiplier

Now let us try a different method. We want to implement the condition $A_1 = A_2$ on Σ by adding a Lagrange multiplier term to the action which enforces the condition. The canonical choice for such a term would be as follows. Take a 1-form $\varphi \in \Omega^1(\Sigma)$ and add the term

$$\frac{i}{4\pi}\int_\Sigma \varphi \wedge (A_1 - A_2) \tag{5.3.1}$$

to the action (5.2.1). However, using this term it is not possible to make the action gauge invariant. In fact, the term itself is not even well-defined since the fields A_j are only defined locally on Σ but not globally (if the bundle is not trivial).

Therefore, it seems to be a better choice to take a 0-form $\varphi : \Sigma \to \mathbb{R}$ and add the term

$$-\frac{i}{2\pi}\int_\Sigma v\, \varphi(dA_1 - dA_2) \tag{5.3.2}$$

to S (v is a parameter specifying the surface operator). The equation of motion for φ gives the constraint

$$dA_1 = dA_2 \qquad \text{on } \Sigma. \tag{5.3.3}$$

Locally (that is, on simply connected open subsets of Σ) this implies that $A_1 - A_2 = df$ for some function f. But this means that A_1 and A_2 only differ by a gauge transformation!

We can now define a sensible gluing condition on Σ by gauging such that $A_1 = A_2$ on some small open subset of Σ and then restricting gauge transformations to those leaving the equation $A_1 = A_2$ invariant. Of course, these gauge transformations are precisely those for which $df_1 = df_2$. This is familiar from (5.2.6).

Note that the action has a symmetry under $\varphi \mapsto \varphi + \frac{2\pi\ell}{v}$. Thus, we may consider φ to be the lift of a map $\tilde{\varphi} : \Sigma \to \mathbb{R}/2\pi x\mathbb{Z}$ if x is chosen such that $vx \in \mathbb{Z}$.

Let us turn to gauge invariance. From $df_1 = df_2$ it follows that only gauge transformations of the form $(f, f+c)$ with $c = \text{const}$ are allowed on Σ. Under such a transformation the bare action (5.2.1) changes by

$$\Delta S = k\frac{i}{2\pi}\int_\Sigma f(dA_1 - dA_2) - k\frac{i}{2\pi}\int_\Sigma c\, dA_2 \tag{5.3.4}$$

(note that $f_2 = f_1 + c$ holds *only* on Σ). The first term can be cancelled by an appropriate choice of transformation law for φ. Let[2] $\varphi \mapsto \varphi + \frac{k}{v}f$. This gives us a term

$$-v\frac{k}{v}\frac{i}{2\pi}\int_\Sigma f(dA_1 - dA_2). \tag{5.3.5}$$

which precisely cancels the first term in (5.3.4). The second term cannot be cancelled by φ. In fact, requiring this term to vanish modulo $2\pi i$ gives us the constraint that $c = 2\pi\frac{a}{b}$ where $b|k$.

A close look at the action shows that we have got some symmetry left. Taking $\varphi \mapsto \varphi + \frac{k}{v}f + 2\pi\frac{\ell}{v}$ for some $\ell \in \mathbb{Z}_v$ and using the fact that the dA_i have integral periods, we see that this transformation law leaves S invariant, too.

We can use the gauge transformation f to get further restrictions on the parameters v and x. Since f is only defined modulo 2π the gauge variation $\delta\varphi$ should be well defined modulo $2\pi x$ if we replace f by $f + 2\pi$. Apparently,

$$\delta\varphi \;\mapsto\; \delta\varphi + 2\pi\frac{k}{v} \tag{5.3.6}$$

under $f_1 \mapsto f + 2\pi$, so we need $2\pi\frac{k}{v} = 2\pi m x$ for some $m \in \mathbb{Z}$. Since $vx \in \mathbb{Z}$, this gives us a decomposition of k into a product of integers: $k = m \cdot (vx)$. In particular, if we make the choice of basis $x = 1$, we get the constraints

$$v \in \mathbb{Z}, \qquad v|k. \tag{5.3.7}$$

A consistent way to solve all this is putting $c = 2\pi\frac{\ell}{v}$ which shows that the gauge group on Σ is

$$\left\{(u, u + 2\pi\tfrac{\ell}{v})\middle| u \in U(1), \ell \in \mathbb{Z}_v\right\} \cong U(1) \times \mathbb{Z}_v \tag{5.3.8}$$

[2] This transformation law for φ shows that it was actually *necessary* to consider φ as a lift of a S^1-valued field, since f is itself such a lift.

Remembering that we can interpret the boundary condition in this theory as a surface operator in a $U(1)$ Chern-Simons theory we see that the surface operator creates a $\mathbb{Z}_v$-orbifold of the original theory. This is in fact another point where the relation to conformal field theory becomes obvious. The conformal field theory created by the surface operator defined by $v \in \mathbb{Z}$ creates a $\mathbb{Z}_v$-orbifold of the theory created by the trivial surface operator. We will see this explicitly in the next section.

We want to rephrase and emphasize the above results as follows.

Given some integer v, the group $U(1) \times \mathbb{Z}_v$ can only appear as the boundary gauge group of the above theory if the level k is a multiple of v.

This insight will be important later.

5.4. The compactified free Boson

A surface operator in $U(1)$ Chern-Simons theory at level k corresponds rational[3] $\hat{u}(1)$ free boson theory (which is the only rational $c = 1$ theory with current algebra $\hat{u}(1)$). The fields of this theory take values on a circle of radius R. For an introduction to the free Boson CFT, see e.g. [15] or [16]. We will now derive a specific representation for the Hilbert space of this theory. In general, the Hilbert space takes the following form

$$\mathcal{H} = \bigoplus_{Q_i, \bar{Q}_i} \mathcal{V}_{Q_i} \otimes \overline{\mathcal{V}}_{\bar{Q}_i}, \tag{5.4.1}$$

where $(Q_i, \bar{Q}_i)$ takes values in an even, self-dual lattice Γ called the *charge lattice*. For a given compactification radius R, this lattice can be parametrized as follows

$$\Gamma = \left\{ \frac{1}{\sqrt{2}} \left(\frac{n}{R} - mR, \frac{n}{R} + mR \right) \,\middle|\, m, n \in \mathbb{Z} \right\}. \tag{5.4.2}$$

Thus, from now on we will use the notation

$$Q_{nm} = \frac{1}{\sqrt{2}} \left(\frac{n}{R} - mR \right) , \quad \bar{Q}_{nm} = \frac{1}{\sqrt{2}} \left(\frac{n}{R} + mR \right) \tag{5.4.3}$$

Obviously, this theory is not rational with respect to $\hat{\mathfrak{u}}(1)$, since we are summing over infinitely many modules. However, if the squared radius R^2 is rational, the theory turns out to be rational with respect to a larger symmetry algebra as we will show now.

[3] A conformal field theory with Hilbert space $\mathcal{H} = \bigoplus_{i,j \in I} N_{ij} \mathcal{V}_i \otimes \bar{\mathcal{V}}_j$ is called rational if I is a finite set.

Suppose, we have $R^2 = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ coprime. Then, Q_{nm} and $\bar{Q}_{nm}$ can be written as

$$Q_{nm} = \frac{1}{\sqrt{2pq}}(nq - mp)\,, \quad \bar{Q} = \frac{1}{\sqrt{2pq}}(nq + mp) \tag{5.4.4}$$

and $\mathcal{H}$ takes the form

$$\mathcal{H} = \bigoplus_{n,m\in\mathbb{Z}} \mathcal{V}_{\frac{1}{\sqrt{2pq}}(nq-mp)} \otimes \overline{\mathcal{V}}_{\frac{1}{\sqrt{2pq}}(nq+mp)}. \tag{5.4.5}$$

Now, let $n = \tilde{n} + pN$ and $m = \tilde{m} + qM$, with $\tilde{n} \in \mathbb{Z}_p, \tilde{m} \in \mathbb{Z}_q$ and $M, N \in \mathbb{Z}$. In terms of these new variables, $\mathcal{H}$ takes the form

$$\mathcal{H} = \bigoplus_{\substack{\tilde{n}\in\mathbb{Z}_p,\tilde{m}\in\mathbb{Z}_q\\ M,N\in\mathbb{Z}}} \mathcal{V}_{\tilde{Q}^{MN}_{\tilde{n}\tilde{m}}} \otimes \overline{\mathcal{V}}_{\tilde{\bar{Q}}^{MN}_{\tilde{n}\tilde{m}}}, \tag{5.4.6}$$

with

$$\tilde{Q}^{MN}_{\tilde{n}\tilde{m}} = \frac{1}{\sqrt{2pq}}(\tilde{n}q - \tilde{m}p + pq(N - M)), \tag{5.4.7}$$

$$\tilde{\bar{Q}}^{MN}_{\tilde{n}\tilde{m}} = \frac{1}{\sqrt{2pq}}(\tilde{n}q + \tilde{m}p + pq(N + M)). \tag{5.4.8}$$

Putting $A = N - M$ and $B = N + M$, we can write this as

$$\mathcal{H} = \bigoplus_{\substack{\tilde{n}\in\mathbb{Z}_{2p}\\ \tilde{m}\in\mathbb{Z}_q\\ A,B\in\mathbb{Z}}} \mathcal{V}_{\frac{1}{\sqrt{2pq}}(\tilde{n}q-\tilde{m}p+2pqA)} \otimes \overline{\mathcal{V}}_{\frac{1}{\sqrt{2pq}}(\tilde{n}q+\tilde{m}p+2pqB)} \tag{5.4.9}$$

$$= \bigoplus_{\tilde{n},\tilde{m}} \underbrace{\left(\bigoplus_{A\in\mathbb{Z}} \mathcal{V}_{\frac{1}{\sqrt{2pq}}(\tilde{n}q-\tilde{m}p+2pqA)}\right)}_{=:\mathcal{V}^{\text{ext}}_{\frac{\tilde{n}q-\tilde{m}p}{\sqrt{2pq}}}} \otimes \underbrace{\left(\bigoplus_{B\in\mathbb{Z}} \overline{\mathcal{V}}_{\frac{1}{\sqrt{2pq}}(\tilde{n}q+\tilde{m}p+2pqB)}\right)}_{=:\overline{\mathcal{V}}^{\text{ext}}_{\frac{\tilde{n}q+\tilde{m}p}{\sqrt{2pq}}}} \tag{5.4.10}$$

$$= \bigoplus_{\tilde{n},\tilde{m}} \mathcal{V}^{\text{ext}}_{\frac{\tilde{n}q-\tilde{m}p}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{\tilde{n}q+\tilde{m}p}{\sqrt{2pq}}}, \tag{5.4.11}$$

where it is understood that we take $\tilde{n}q - \tilde{m}p$ modulo $2pq$. Now, we have managed to write $\mathcal{H}$ as a sum over *finitely* many modules. We call these new modules the *extended* modules since they have a larger symmetry algebra than the ones we began with. With respect to this algebra, the theory is rational. Note that in the above we have $\tilde{n}q - \tilde{m}p \in \mathbb{Z}_{2pq}$ and $\tilde{n}q + \tilde{m}p \in \mathbb{Z}_{2pq}$

We can perform one more manipulation to bring the Hilbert space into a nicer form. Since p and q are coprime, the Bézout identity tells us that we can find integers M, N such that $qM - pN = 1$. Now, put

$$\alpha := qM + pN \pmod{2pq}. \tag{5.4.12}$$

A quick calculation shows that $\alpha(qm-pn) = qm+pn \pmod{2pq}$ and $\alpha^2 = 1 \pmod{2pq}$. Thus, multiplication by α is an automorphism of $\mathbb{Z}_{2pq}$ and we can write

$$\mathcal{H} = \bigoplus_{a\in\mathbb{Z}_{2pq}} \mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{\alpha a}{\sqrt{2pq}}}. \tag{5.4.13}$$

We can already smell the relationship to Chern-Simons theory. The group $\mathbb{Z}_{2pq}$ appearing here labels the different vacua of the theory, or equivalently, the different primary vertex operators. Now recall that in section 2.3 we mentioned that primary fields in CFT correspond to line operators in Chern-Simons theory. These are classified by the discriminant group (see section 3.1) of abelian Chern-Simons theory which for given level k happens to be $D = \mathbb{Z}_{2k}$ (see section 3.1.1).

Thus, p and q are divisors of k specifying the CFT which belongs to some surface operator. But this situation is familiar! In section 5.1, we learned that surface operators in $U(1)$ Chern-Simons theory are characterized by two integers v and w dividing k.

Remark: In the above computations, we took p and q to be coprime. This is convenient but not necessary. We could as well have given them some common factor α. For the CFT, p and q not being coprime means losing symmetry. Indeed, if we replace $p \mapsto \alpha p$ and $q \mapsto \alpha q$ in (5.4.9), we see that the extended modules become smaller and can therefore shelter only a representation of a smaller symmetry algebra.

5.4.1. $\mathbb{Z}_q$ Orbifold of the free Boson

In section 5.3 we mentioned that a surface operator in $U(1)$ Chern-Simons theory creates a $\mathbb{Z}_v$ orbifold of the theory. From the correspondence described in the previous section we expect to be able to see this in the two-dimensional theory. Indeed, we will now show that the free Boson theory defined by two integers p, q with $pq = k$ is a $\mathbb{Z}_q$ orbifold of the theory defined by $q = 1, p = k$ (which corresponds to the trivial surface operator).

To see this, consider the Hilbert space (5.4.13) for $q = 1, p = k$. In this case, we have $nq - mp = nq + mp \pmod{2k}$ and so $\mathcal{H}$ is the diagonal Hilbert space

$$\mathcal{H} = \bigoplus_{a\in\mathbb{Z}_{2k}} \mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{a}{\sqrt{2k}}}. \tag{5.4.14}$$

On this space, define a $\mathbb{Z}_q$-action as follows. Let the generator ρ act via

$$\rho(x\otimes y) = e^{-\frac{2\pi i}{2k}(a-\alpha b)} x\otimes y \qquad \text{for} \quad x\otimes y \in \mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b}{\sqrt{2k}}}, \tag{5.4.15}$$

where α is the automorphism of $\mathbb{Z}_{2k}$ defined in (5.4.12). That this is indeed a $\mathbb{Z}_q$-action on $\mathcal{H}$ can be seen as follows. For $x \otimes y$ in the diagonal $\mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{a}{\sqrt{2k}}}$, the action of ρ

becomes

$$\rho = e^{-\frac{2\pi i}{2k}(1-\alpha)a}. \tag{5.4.16}$$

Now, remember that we had $\alpha = qM + pN$ with $qM - pN = 1$ in $\mathbb{Z}_{2k}$. Thus, we have

$$\begin{aligned} 1 - \alpha &= qM - pN - (qM + pN) \\ &= -2pN. \end{aligned}$$

This shows that if we apply ρ q times, we get the identity operator

$$\begin{aligned} \rho^q &= e^{-\frac{2\pi i}{2k}(1-\alpha)aq} \\ &= e^{2\pi i \frac{2Napq}{2k}} \\ &= 1, \end{aligned}$$

since $pq = k$.

We are going to determine the orbifold of this theory with respect to this group action. To do this, we will first have to determine the twisted sectors introduced by the action and then sort out those which transform trivially.

Twisted sectors. To get the twisted sectors, consider the ℓ-twisted torus partition function with ρ^ℓ inserted

$$\mathrm{Tr}_{\mathcal{H}}(q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \rho^\ell) \tag{5.4.17}$$

This is nothing but the S transform of the partition function of the ℓ-twisted Hilbert space

$$Z_\ell = \mathrm{Tr}_{\mathcal{H}_\ell}(q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}}), \tag{5.4.18}$$

where $\mathcal{H}_\ell$ denotes the ℓ-twisted Hilbert space. Thus, we have

$$\begin{aligned} Z_\ell &= S\Big(\mathrm{Tr}_{\mathcal{H}}(q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \rho^\ell)\Big) \\ &= S\Big(\sum_{a \in \mathbb{Z}_{2k}} \chi_a(q) \bar{\chi}_a(\bar{q}) e^{-\frac{2\pi i}{2k}(1-\alpha)a}\Big). \end{aligned}$$

The S-transformation of one single character χ_a has been computed in [17] to be

$$S(\chi_a(\tau)) = \frac{1}{\sqrt{2k}} \sum_{b \in \mathbb{Z}_{2k}} e^{\frac{2\pi i}{2k} ab} \chi_b(\tau). \tag{5.4.19}$$

Thus, we have

$$\begin{aligned} S\Big(\sum_{a \in \mathbb{Z}_{2k}} \chi_a(q) \bar{\chi}_a(\bar{q}) e^{\frac{2\pi i}{2k}(\alpha - 1)a\ell}\Big) &= \frac{1}{2k} \sum_a \sum_{b,c} e^{2\pi i \frac{a}{2k}(b-c)} e^{-2\pi i \frac{\ell(1-\alpha)a}{2k}} \chi_b(\tau) \bar{\chi}_c(\bar{\tau}) \\ &= \frac{1}{2k} \sum_a \sum_{b,c} e^{2\pi i \frac{a}{2k}(b - c - \ell(1-\alpha))} \chi_b(\tau) \bar{\chi}_c(\bar{\tau}). \end{aligned}$$

Using the identity $\sum_{a\in\mathbb{Z}_{2k}} e^{2\pi i \frac{a}{2k} b} = 2k\,\delta_{b0}$, this becomes

$$\sum_b \chi_b\,\bar{\chi}_{b-\ell(1-\alpha)}.$$

Hence, the twisted Hilbert space is

$$\mathcal{H}_\ell = \bigoplus_{a\in\mathbb{Z}_{2pq}} \mathcal{V}^{\text{ext}}_{\frac{b}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b-\ell(1-\alpha)}{\sqrt{2pq}}}. \tag{5.4.20}$$

Sorting out the vacuum representation. Of all these twisted Hilbert spaces, we now project onto those which are invariant under our $\mathbb{Z}_q$-action. For $\mathcal{V}^{\text{ext}}_{\frac{b}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b-\ell(1-\alpha)}{\sqrt{2pq}}}$ to transform trivially under ρ, one needs $b-\alpha(b-(1-\alpha)\ell)=0)$, i.e. $(1-\alpha)\ell=(1-\alpha)b$ (recall that $\alpha^2=1$). Thus, the Hilbert space of the orbifold theory is

$$\mathcal{H}_{\text{orb}} = P\Big(\bigoplus_{\ell\in\mathbb{Z}_q}\mathcal{H}_\ell\Big) \tag{5.4.21}$$

$$= \bigoplus_{b\in\mathbb{Z}_{2k}} \mathcal{V}^{\text{ext}}_{\frac{b}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b-(1-\alpha)b}{\sqrt{2pq}}} \tag{5.4.22}$$

$$= \bigoplus_{b\in\mathbb{Z}_{2k}} \mathcal{V}^{\text{ext}}_{\frac{b}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{\alpha b}{\sqrt{2pq}}}, \tag{5.4.23}$$

where P denotes the projection on the invariant subspace. Comparing this to (5.4.13), we see that the orbifold procedure gave us precisely the same result as the surface operator characterized by $q\in\mathbb{Z}$.

5.5. The Fusion of Surface Operators

As we have seen, in some theories surface operators can be described by adding a Lagrange multiplier term to the action which implements the gluing condition. This turns out to be quite convenient since these surface operators can be studied very explicitly. However, there are only few examples where this description is possible, such as abelian Chern-Simons theory. Here, the existence and monoidal structure have successfully been described using Lagrange multipliers by Kapustin and Saulina [6, 1]. Parts of their results were reviewed in section 4.2. They found that surface operators in $U(1)$ Chern-Simons theory can essentially be characterized by divisors of the level k and showed that if v and v' are divisors of k with corresponding surface operators S_v and $S_{v'}$, then their fusion is given by the following formula

$$S_v \circ S_{v'} = GS_{v''}, \tag{5.5.1}$$

where $G = \gcd(v, v', \frac{k}{v}, \frac{k}{v'})$ and $v'' = \mathrm{lcm}(\gcd(v, \frac{k}{v'}), \gcd(v', \frac{k}{v}))$.

We can get a glimpse of the rules for fusing surface operators from the above considerations. However, it does not seem possible to obtain the full fusion law (5.5.1) with our purely classical considerations.

But let us see how far we get. Suppose we have two surface operators supported on surfaces Σ_1 and Σ_2. Let us call the pieces into which M is divided M_1, M_2 and M_3. The surface operators will impose the constraints

$$f_2 = f_1 + 2\pi\frac{\ell}{v}, \quad f_3 = f_2 + 2\pi\frac{\ell'}{v'}. \tag{5.5.2}$$

If we fuse the surfaces Σ_1 and Σ_2, the above equations must hold on the merged surface and we can eliminate f_2 from them to obtain

$$\begin{aligned} f_3 &= f_1 + 2\pi\frac{\ell}{v} + 2\pi\frac{\ell'}{v'}. \\ &= f_1 + 2\pi\frac{\ell v' + \ell' v}{vv'} \\ &= f_1 + 2\pi\frac{\ell'' \gcd(v, v')}{vv'} \\ &= f_1 + 2\pi\frac{\ell''}{\mathrm{lcm}(v, v')}, \end{aligned}$$

where we have used Bezout's identity in the third line. This suggests that the fusion of two surface operators characterized by the integers v and v' is the surface operator belonging to $\mathrm{lcm}(v, v')$.

We know from (5.5.1) that this cannot be the whole truth. It is simply too weak a restriction as we can see by noting that $\mathrm{lcm}(\gcd(v, k/v'), \gcd(v', k/v))$ is a divisor of $\mathrm{lcm}(v, v')$ and therefore

$$\left\{\frac{\ell}{\mathrm{lcm}(\gcd(v, k/v'), \gcd(v', k/v))} \pmod 1\right\} \subset \left\{\frac{\ell}{\mathrm{lcm}(v, v')} \pmod 1\right\} \tag{5.5.3}$$

so the actually correct values appear among those found by us.

In the next section, we will attempt to rederive formula (5.5.1) from a different point of view.

As indicated in sections 2.3 and 3.2, surface operators in Chern-Simons theory create two-dimensional conformal field theories. It suggests itself that one should be able to describe the fusion of two surface operators in two-dimensional terms. In the following we will do this for $U(1)$ Chern-Simons theory and compare our results to those in [6].

5.6. Fusion from the two-dimensional point of view

5.6.1. p, q Coprime

The partition function of the above theory takes the form

$$\begin{aligned} Z &= \sum_{a\in\mathbb{Z}_{2pq}} \chi_a(q)\bar{\chi}_{\alpha a}(\bar{q}), \qquad q = e^{2\pi i\tau} \\ &= \sum_{a,b\in\mathbb{Z}_{2pq}} \delta_{b,\alpha a}\chi_a(q)\bar{\chi}_b(\bar{q}). \end{aligned}$$

for which we can define the coefficient matrix $N_{ab} := \delta_{b,\alpha a}$.

Now, if we take two surface operators in the three-dimensional theory and fuse them, the coefficient matrix of the CFT corresponding to the fused surface operator[4] is expected to be the product of the coefficient matrices of the two former CFTs [18]. Let us compute that.

We take a free boson CFT with radius $R = \sqrt{\frac{p}{q}}$ with p, q coprime and $pq = k$ and another one with radius $R' = \sqrt{\frac{p'}{q'}}$ with p', q' coprime and $p'q' = k$ (note that both CFTs have the same k).

Choose M, N, M', N' such that $qM - pN = q'M' - p'N' = 1$ and put $\alpha = qM + pN$ $\beta = q'M' + p'N'$. The product of the coefficient matrices is

$$N_{ac}N'_{cb} = \delta_{\alpha a,c}\delta_{\beta c,b} = \delta_{\beta\alpha a,b}. \tag{5.6.1}$$

To get the radius for the fused CFT, we have to bring the product $\beta\alpha$ into the form

$$\beta\alpha = q''M'' + p''N'' \tag{5.6.2}$$

with M'', N'' satisfying $q''M'' - p''N'' = 1$. We calculate

$$\begin{aligned} \beta\alpha &= (q'M' + p'N')(qM + pN) \\ &= qq'MM' + pp'NN' + qp'MN' + pq\text{"}NM' \\ &= qq'MM' + \frac{k}{q}\frac{k}{q'}NN' + q\frac{k}{q'}MN' + \frac{k}{q}q'NM', \end{aligned}$$

where in the last line we have eliminated the p's to make a connection in notation to Chern-Simons theory. Now, we define

$$q'' := \mathrm{lcm}\big(\gcd(q, \tfrac{k}{q'}), \gcd(q', \tfrac{k}{q})\big) \tag{5.6.3}$$

[4] Henceforth referred to as the *fused CFT*.

and

$$M'' := \tfrac{1}{q''}\left(qq'MM' + \tfrac{k}{q}\tfrac{k}{q'}NN'\right) \tag{5.6.4}$$

$$N'' := \tfrac{q''}{k}\left(q\tfrac{k}{q'}MN' + \tfrac{k}{q}q'NM'\right) \tag{5.6.5}$$

With these definition, the equation $\beta\alpha = q''M'' + \frac{k}{q''}N''$ is manifest. We need to check:

1. M'' is an integer
2. N'' is an integer
3. The above defined quantities satisfy $q''M'' - \frac{k}{q''}N'' = 1$

Proof.

1. From the general formula $\mathrm{lcm}(a,b)\cdot\gcd(a,b) = a\cdot b$ it follows that

$$q'' = \frac{\gcd(q,\frac{k}{q'})\gcd(q',\frac{k}{q})}{\gcd(\gcd(q,\frac{k}{q'}),\gcd(q',\frac{k}{q}))}. \tag{5.6.6}$$

   The denominator is 1, since q and $\frac{k}{q}$ were assumed to be coprime. Now note that the product $\gcd(q,\frac{k}{q'})\gcd(q',\frac{k}{q})$ is contained as a factor in qq' as well as in $\frac{k}{q}\frac{k}{q'}$, so one can take it out of the parenthesis and M'' stays an integer.

2. We consider the first term in N'':

$$\frac{q''}{k}q\frac{k}{q'} = q''\frac{q}{q'}. \tag{5.6.7}$$

   It suffices to show that this an integer (the treatment of the second term is analogous). We have

$$\begin{aligned} q''\frac{q}{q'} &= \frac{q}{q'}\frac{\gcd(q,\frac{k}{q'})\gcd(q',\frac{k}{q})}{\gcd(\gcd(q,\frac{k}{q'}),\gcd(q',\frac{k}{q}))} \\ &= \frac{q}{q'}\frac{q'\frac{k}{q}}{\mathrm{lcm}(q',\frac{k}{q})}\frac{\gcd(q,\frac{k}{q'})}{\gcd(q,\frac{k}{q'},q',\frac{k}{q})} \\ &= \frac{k}{\mathrm{lcm}(q',\frac{k}{q})}\cdot\frac{\gcd(q,\frac{k}{q'})}{\gcd(q,\frac{k}{q'},q',\frac{k}{q})} \end{aligned}$$

   The second factor is manifestly an integer. But the first one is an integer, too, since q' and $\frac{k}{q}$ divide k an therefore $\mathrm{lcm}(q',\frac{k}{q})$ divides k.

3. This is a straightforward calculation. We have

$$\begin{aligned} q''M'' - \tfrac{k}{q''}N'' &= qq'MM' + \frac{k}{q}\frac{k}{q'}NN' - q\frac{k}{q'}MN' - \frac{k}{q}q'NM' \\ &= qM(q'M' - \tfrac{k}{q'}N') + \tfrac{k}{q}N(\tfrac{k}{q'}N' - q'M') \\ &= qM - \tfrac{k}{q}N \\ &= 1 \end{aligned}$$

□

The result (5.6.3) coincides precisely with that of Kapustin and Saulina in [6] (compare to equation (5.5.1) and note that $G = 1$ if v and $\frac{k}{v}$ are coprime).

In total, the Hilbert space of the fused theory reads

$$\mathcal{H}'' = \bigoplus_{a,b\in\mathbb{Z}_{2k}} \delta_{b,\beta\alpha a} \mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2pq}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b}{\sqrt{2pq}}}. \tag{5.6.8}$$

5.6.2. The non-coprime case

The result (5.6.3) is nice, but somewhat unsatisfactory. Although it agrees with (5.5.1), it only covers the case of the divisors of k being coprime. But from the three-dimensional point of view there is no apparent reason why this restriction should be made and indeed, formula (5.5.1) holds for any two divisors of k, regardless of their greatest common divisor. Thus, we should be able to derive this formula also if p and q are not coprime. So let us start again in the following setting. We want to fuse two surface operators in $U(1)$-Chern-Simons theory at level k whose related CFTs are CFT_1 and CFT_2, respectively. We take CFT_1 to be specified by the integers p, q with $g := \gcd(p, q)$ not necessarily equal to one and CFT_2 to be specified similarly by p', q', g'. Since both CFTs belong to Chern-Simons theories at level k, we require $pq = p'q' = k$.

The Hilbert spaces of both CFTs still take the form (5.4.11). In particular, we have $nq - mp \in \mathbb{Z}_{2k}$ and $nq' - mp' \in \mathbb{Z}_{2k}$. The difference to the case before is that now $nq - mp$ is always a multiple of g and $nq' - mp'$ is always a multiple of g'. This has the consequence that now we cannot construct an automorphism α anymore which sends $nq - mp$ to $nq + mp$ and thus it is not as easy as before to write the Hilbert spaces in a nice form with a coefficient matrix N_{ab}.

Resolving redundancy. Let us start by working out the form of the coefficient matrix of CFT_1. That is, we want to find a matrix N_{ab} such that we can write

$$\mathcal{H} = \bigoplus_{\substack{n\in\mathbb{Z}_{2p}\\ m\in\mathbb{Z}_q}} \mathcal{V}^{\text{ext}}_{\frac{nq-mp}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{nq+mp}{\sqrt{2k}}} \tag{5.6.9}$$

$$= \bigoplus_{a,b\in\mathbb{Z}_{2k}} N_{ab}\, \mathcal{V}^{\text{ext}}_{\frac{a}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}^{\text{ext}}_{\frac{b}{\sqrt{2k}}}. \tag{5.6.10}$$

Let $a = nq - mp \in \mathbb{Z}_{2k}$ be fixed. If p and q were coprime, this would fix m and n uniquely. But now that $g \neq 1$ is possible, we have to find out how many m, n there are such that $nq - mp = a$.

So let m_0, n_0 be fixed such that $n_0 q - m_0 p = a$. It is clear that a transformation

$$n_0 \mapsto n_0 + x, \qquad m_0 \mapsto m_0 + y \tag{5.6.11}$$

leaves a invariant if and only if

$$xq - yp = 0 \qquad (\text{mod}\, 2k) \tag{5.6.12}$$

(of course, we take $x \in \mathbb{Z}_{2p}, y \in \mathbb{Z}_q$). This is equivalent to

$$\begin{aligned} &xq - yp = 2pqv \quad \text{for some } v \in \mathbb{Z} \\ \Rightarrow\ &q \text{ divides } yp \\ \Rightarrow\ &q \text{ divides } yg \end{aligned}$$

This suggests to put $y = w\frac{q}{g} =: w\tilde{q}$ with $w \in \mathbb{Z}_g$. Then we have

$$\begin{aligned} xq &= 2pqv + w\tilde{q}p \\ \Leftrightarrow\ x &= 2pv + w\frac{p}{g} \\ &\equiv w\frac{p}{g} \\ &=: w\tilde{p} \end{aligned}$$

This takes us a big step forward. The pairs (n, m) leaving a invariant are exactly given by $(n_0 + w\tilde{p}, m_0 + w\tilde{q})$ with $w \in \mathbb{Z}_g$.

Of course, the same reasoning can be applied to our second CFT to obtain $(x', y') = (w'\tilde{p}', w'\tilde{q}')$ with $w' \in \mathbb{Z}_{g'}$.

Matrix product. To determine the matrix product $N_{ac}N'_{cb}$, we have to determine all c belonging to the same a. That is, we have to find all $nq+mp$ if $nq-mp=a$. But this is a simple calculation. Let $a=n_0q-m_0p$ and denote $c_0=n_0q+m_0p$. Then we have

$$a=(n_0+w\tilde{p})q-(m_0+w\tilde{q})p \rightsquigarrow c=(n_0+w\tilde{p})q+(m_0+w\tilde{q})p$$
$$=c_0+w\frac{2k}{g}.$$

This makes it clear that the matrix N_{ab} will not have the form $\delta_{b,\alpha a}$ anymore but will rather be a *sum* of Kronecker symbols.

Homomorphism α. We already noted that it is not possible anymore to find an automorphism α of $\mathbb{Z}_{2k}$ sending $nq-mp$ to $nq+mp$. However, we can do something which is almost as good. Since g is the greatest common divisor of p and q, we can find integers M,N such that $Nq-Mp=g$. If we now define

$$\bar{\alpha}:=Nq+Mp, \tag{5.6.13}$$

we obtain a homomorphism of the subgroup $\mathbb{Z}_{\frac{2k}{g}}$ of $\mathbb{Z}_{2k}$ consisting of multiples of g. This homomorphism has the property

$$\bar{\alpha}(nq-mp)=g(nq+mp) \pmod{2k}. \tag{5.6.14}$$

Note that this is actually a multiple of g^2. Let us denote by $\mathbb{Z}_{\frac{2k}{g^2}}$ the subgroup of $\mathbb{Z}_{2k}$ consisting of multiples of g^2. Now define

$$D_g:\mathbb{Z}_{\frac{2k}{g^2}}\to\mathbb{Z}_{2k};$$
$$x\mapsto\tfrac{x}{g}$$

and

$$\alpha:=D_g\circ\bar{\alpha} \tag{5.6.15}$$

This way we obtain a homomorphism with the desired property $\alpha(nq-mp)=(nq+mp)$. The price we have paid is that α does not hit every element of the form gx anymore, but only those of the form $D_g(g^2x)$. As we will see, this is not a problem.

Using the map α, we are finally able to write the coefficient matrix as[5]

$$N_{ac}=\sum_{w\in\mathbb{Z}_g}\delta_{c,\alpha a+w\frac{2k}{g}}. \tag{5.6.16}$$

We see that the sum over w exactly takes care of the fact that α does not hit all elements of the form gx.

[5] It is understood that $N_{ac}=0$ whenever a or c are not multiples of g.

Matrix product again. Now that we have an explicit form for the coefficient matrix, the rest of the calculation is straightforward. The product of N and N' is

$$N''_{ab} := N_{ac}N'_{cb} = \sum_{c} \sum_{\substack{w\in\mathbb{Z}_g \\ w'\in\mathbb{Z}_{g'}}} \delta_{c,\alpha a + w\frac{2k}{g}} \delta_{b,\beta c + w'\frac{2k}{g'}}, \tag{5.6.17}$$

where $\beta = D_{g'} \circ \bar{\beta}$ is defined similar to α in terms of q', p'. The product of deltas is nonzero only if

$$a, b \in \mathbb{Z}_{\frac{2k}{g}} \cap \mathbb{Z}_{\frac{2k}{g'}} \cong \mathbb{Z}_{\frac{2k}{g''}}, \tag{5.6.18}$$

where $g'' = \text{lcm}(g, g')$, and

$$\begin{aligned} b &= \beta\left(\alpha a + w\frac{2k}{g}\right) + w'\frac{2k}{g'} \\ &= \beta\alpha a + \beta w\frac{2k}{g} + w'\frac{2k}{g'} \end{aligned}$$

So we can rewrite (5.6.17) as

$$N''_{ab} = \sum_{\substack{w\in\mathbb{Z}_g \\ w'\in\mathbb{Z}_{g'}}} \delta_{b,\beta\alpha a + \beta w\frac{2k}{g} + w'\frac{2k}{g'}}. \tag{5.6.19}$$

It is clear from the definition of β that it does not make a difference weather we let the sum run over w or over βw, so we can write

$$N''_{ab} = \sum_{\substack{w\in\mathbb{Z}_g \\ w'\in\mathbb{Z}_{g'}}} \delta_{b,\beta\alpha a + w\frac{2k}{g} + w'\frac{2k}{g'}}. \tag{5.6.20}$$

Redundancy again. Since the index of the Kronecker symbol lies in a cyclic group, we can again remove some redundancy. The summand in (5.6.20) does not change if

$$w\frac{2k}{g} + w'\frac{2k}{g'} = 0 \qquad (\text{mod}\, 2k) \tag{5.6.21}$$

We can use the same reasoning as the one following (5.6.12) to quantify the redundancy.

$$\begin{aligned} w\frac{2k}{g} + w'\frac{2k}{g'} = 2kv \;&\Rightarrow\; wg' + w'g = vgg' \\ &\Rightarrow\; g' \text{ divides } Gw', \end{aligned}$$

where $G = \gcd(g, g') = \gcd(p, q, p', q')$. So let us put $Gw' =: ug'$ for suitable $u \in \mathbb{Z}_G$. Inserting this, we get

$$\begin{aligned} wg' + u\frac{g'}{G}g &= vgg' \\ \Rightarrow \; wg' &= vgg' - u\frac{gg'}{G} \\ &\equiv -u\frac{gg'}{G} \\ &= -ug'' \end{aligned}$$

Because of this $\mathbb{Z}_G$-redundancy, we might as well include a factor G in (5.6.20) and let the sums run over a smaller domain:

$$N''_{ab} = \sum_{w \in \mathbb{Z}_{\frac{g}{G}}} \sum_{w' \in \mathbb{Z}_{\frac{g'}{G}}} G \cdot \delta_{b, \beta\alpha a + w\frac{2k}{g} + w'\frac{2k}{g'}}. \tag{5.6.22}$$

The final manipulation we can make to bring N'' into a nice form is analogous to what we did to get from (5.4.11) to (5.4.13). We write

$$w\frac{2k}{g} + w'\frac{2k}{g'} = \left(w\frac{g'}{G} + w'\frac{g}{G}\right)\frac{2k}{g''} \tag{5.6.23}$$

and note that the first factor on the right hand side is defined modulo g'', so instead of summing over all w, w' we might as well include a term $w''\frac{2k}{g''}$ and sum over all $w'' \in \mathbb{Z}_{g''}$. This brings N'' into its final form

$$\boxed{N''_{ab} = \sum_{w'' \in \mathbb{Z}_{g''}} G\delta_{b, \beta\alpha a + w''\frac{2k}{g''}}} \tag{5.6.24}$$

Finally, we note that the product $\beta\alpha$ is defined on the subgroup $\mathbb{Z}_{\frac{2k}{g''}} \subset \mathbb{Z}_{2k}$. Since it is essentially given by the same numbers as in the case when p and q were coprime, an analogous calculation to the one in section 5.6.1 gives the same expression for q'' as before. Furthermore, it was proven in the appendix of [6] that one indeed has $\gcd\left(q'', \frac{k}{q''}\right) = g''$ which is needed for the above expressions to be consistent.

The result (5.6.24) of course confirms that of Kapustin and Saulina (5.5.1). We obtained the very same integers classifying the fused CFT as they did to classify the fused surface operator.

Extended modules. The extended modules of the fused CFT take the form

$$\mathcal{V}''^{\text{ext}}_{\frac{nq''-mp''}{\sqrt{2k}}} = \bigoplus_{A \in \mathbb{Z}} \mathcal{V}_{\frac{1}{\sqrt{2k}}(nq''-mp''+2kA)} \tag{5.6.25}$$

and an analogous expression for $\overline{\mathcal{V}}''^{\text{ext}}$. We are finally able to write down the Hilbert space of the fused theory which is

$$\mathcal{H}'' = \bigoplus_{a,b\in\mathbb{Z}_{2k}} N''_{ab}\, \mathcal{V}''^{\text{ext}}_{\frac{a}{\sqrt{2k}}} \otimes \overline{\mathcal{V}}''^{\text{ext}}_{\frac{b}{\sqrt{2k}}}. \tag{5.6.26}$$

Remarks: We have seen that the fusion of Surface Operators in Chern-Simons theory leads to a pairing which takes two CFTs and turns them into a new one. One might hope to obtain a nice product structure on the space of CFTs. However, there is the issue that the product obtained above depends not only on the Hilbert space of the conformal field theory, but also on the way we decompose it into extended modules. This is not surprising, since the CFT does not determine the Chern-Simons theories on both sides of the surface operator. Under a rescaling $(p,q) \mapsto (\alpha p, \alpha q)$ the radius of the CFT stays the same, but the level of the corresponding Chern-Simons theory changes as $k \mapsto \alpha^2 k$.

But, of course, a product on the space of CFTs should only depend on the CFT itself and not on the way we write it down.

In fact, to the author's knowledge there is no way to turn the above assignment into a sensible product structure. The most promising starting point for doing so seems do be defining the product for non-rational CFTs (since this avoids the appearance of certain infinities). However, this attempt fails. The basic problems are outlined in Appendix B.

5.7. Wilson lines and Primary Operators

In this last subsection we want to investigate how the discrete part of the gauge group in (4.3.2) can be seen in the two-dimensional theory by studying the transformation behaviour of certain operators under this group.

We have already seen that a surface operator in $U(1)$ Chern-Simons theory creates a $\hat{\mathfrak{u}}(1)$-CFT. A particular feature of this correspondence which has been known for some time is that primary operators in the CFT correspond to Wilson lines in Chern-Simons theory which pierce the surface operator [3, 6].

Let us first investigate the Symmetry in the CFT. The decomposition (5.4.11) of the Hilbert space makes it clear that the antidiagonal $U(1)$

$$U_{pq} := \exp\left(2\pi i \frac{Q-\bar{Q}}{\sqrt{2pq}}\right) \tag{5.7.1}$$

(where the Q's are given by (5.4.4)) creates a $\mathbb{Z}_q$-symmetry of the theory (note that this is precisely the same operator as the one in section 5.4.1). Explicitly, this operator acts on a state $|Q\rangle \otimes |\bar{Q}\rangle$ by

$$|Q\rangle \otimes |\bar{Q}\rangle \mapsto \exp\left(2\pi i \tfrac{Q}{\sqrt{2pq}}\right)|Q\rangle \otimes \exp\left(-2\pi i \tfrac{\bar{Q}}{\sqrt{2pq}}\right)|\bar{Q}\rangle \tag{5.7.2}$$

Remembering that the vacua of the Fock modules can be obtained from the absolute vacuum by applying a vertex operator $|Q\rangle = V_Q(0)|0\rangle$, we see that on vertex operators $V_Q \otimes V_{\bar{Q}}$, U_{pq} acts via

$$\begin{aligned} V_Q \otimes V_{\bar{Q}} &\mapsto \exp\left(2\pi i \tfrac{Q}{\sqrt{2pq}}\right) V_Q \otimes \exp\left(-2\pi i \tfrac{\bar{Q}}{\sqrt{2pq}}\right) V_{\bar{Q}} && (5.7.3)\\ &= \exp\left(2\pi i \frac{Q-\bar{Q}}{\sqrt{2pq}}\right) V_Q \otimes V_{\bar{Q}} && (5.7.4)\\ &= \exp\left(-2\pi i \frac{m}{q}\right) V_Q \otimes V_{\bar{Q}}. && (5.7.5)\end{aligned}$$

Clearly, applying this operator q times gives the identity. Therefore, we have identified a $\mathbb{Z}_q$-symmetry in the theory which acts on vertex operators by multiplication with $\exp\left(-2\pi i \frac{m}{q}\right)$.

Now let us turn to the three-dimensional theory. From the previous considerations we expect that $q \leftrightarrow v$ and $p \leftrightarrow \frac{k}{v}$. In other words, this means that the trivial surface operator (that is, $v = 1$) corresponds to the diagonal free Boson CFT. In fact, if we put $q = 1$, we have $\sqrt{2k}Q = n + mk$ and $\sqrt{2k}\bar{Q} = n - mk$ which are equal modulo $2k$. Therefore, the representation (5.4.11) is diagonal in this case.

With this expectation in mind, let us see how Wilson lines transform under the discrete part of the boundary gauge group. Let

$$W_X(A) = \exp\left(\int_\gamma X(A)\right) = \exp\left(i\int_{\gamma_1} X_1 A_1 + i\int_{\gamma_2} X_2 A_2\right) \tag{5.7.6}$$

be a Wilson line. In our case, X_1, X_2 are integers (see section 3.1 for details) and γ_1 and γ_2 are curves meeting at a point $x \in \Sigma$.

Now, let f_1, f_2 be gauge transformations such that $(f_1|_\Sigma, f_2|_\Sigma) = (0, 2\pi\frac{\ell}{v})$ (i.e. taking values in the discrete part only) and let A' be the gauge transformed connection. Then, near the point x, we have

$$\begin{aligned} \int_\gamma X(A) - \int_\gamma X(A') &= \int_{\gamma_1} X_1 df_1 + \int_{\gamma_2} X_2 df_2 \\ &= X_1 f_1(x) + X_2 f_2(x) \\ &= X_2\, 2\pi\frac{\ell}{v} \end{aligned}$$

Therefore, a Wilson line with charge (X_1, X_2) transforms as

$$W_X(A) \mapsto W_X(A)\, e^{X_2 2\pi i \frac{\ell}{v}} \tag{5.7.7}$$

This is the three-dimensional analogue to (5.7.5). We see that indeed $v \leftrightarrow q$ and we can even see finer details of the correspondence $W_X(A) \leftrightarrow V_Q \otimes V_{\bar{Q}}$: both operators are characterized by a pair of integers (n, m in the 2d case and X_1, X_2 in the 3d case) and the transformation law (5.7.7) shows that $X_2 \leftrightarrow m$. At first glance, the number ℓ in (5.7.7) does not seem to have an obvious counterpart in the 2d theory, but of course this corresponds to the number of times we apply the operator U_{pq}.

6. The non-abelian Case

Now we will take up the issue of boundary conditions in non-abelian Chern-Simons theory .

6.1. The boundary term

First we compute the boundary contribution to the equation of motion. Recall that the Chern-Simons action in the non-abelian case reads

$$S = \frac{k}{8\pi^2} \int_M \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big) \qquad (\mathrm{mod}\ 1). \tag{6.1.1}$$

The variation of the action is straightforwardly computed as follows. Take a path A_s of connections and note that for each t the collection of all A_s forms a connection $\overline{A}$ on the cylinder $[0,t] \times M$ whose curvature is $\overline{F} = F_s + ds \wedge \dot{A}_s$ (where F_s denotes the curvature of A_s).

Since $[0,t] \times M$ bounds the manifold $(\{0\} \times M) \cup (\{t\} \times M) \cup ([0,t] \times \partial M)$, we have

$$S(A_t) - S(A_0) + S\big(\overline{A}|_{[0,t]\times\partial M}\big) = S\big(\overline{A}|_{\partial([0,t]\times M)}\big) \tag{6.1.2}$$

$$= \frac{k}{8\pi^2} \int_{[0,t]\times M} \overline{F} \wedge \overline{F} \tag{6.1.3}$$

from which we compute the variation to be

$$\delta S = \frac{d}{dt}\Big|_{t=0} S(A_t) = \frac{d}{dt}\Big|_{t=0} S\big(\overline{A}|_{\partial([0,t]\times M)}\big) - \frac{d}{dt}\Big|_{t=0} S\big(\overline{A}|_{[0,t]\times\partial M}\big) \tag{6.1.4}$$

$$= \frac{k}{8\pi^2} \frac{d}{dt}\Big|_{t=0} \int_{[0,t]\times M} \overline{F} \wedge \overline{F} \tag{6.1.5}$$

$$- \frac{k}{8\pi^2} \frac{d}{dt}\Big|_{t=0} \int_{[0,t]\times\partial M} \mathrm{Tr}\big(A_s \wedge d(A_s) + \frac{2}{3} A_s \wedge A_s \wedge A_s\big) \tag{6.1.6}$$

$$= 2\frac{k}{8\pi^2} \frac{d}{dt}\Big|_{t=0} \int_0^t ds \int_M \mathrm{Tr}\big(F_s \wedge \dot{A}_s\big) \tag{6.1.7}$$

$$- \frac{k}{8\pi^2} \frac{d}{dt}\Big|_{t=0} \int_{[0,t]\times\partial M} \mathrm{Tr}\big(A_s \wedge [(d|A)_s + \dot{A}_s \wedge ds]\big), \tag{6.1.8}$$

where $d|$ denotes the exterior derivative on ∂M and the last term in (6.1.6) vanishes for dimensional reasons since for each s the form $A_s \wedge A_s \wedge A_s$ is a 3-form on ∂M. So we have

$$\delta S = 2\frac{k}{8\pi^2}\int_M \mathrm{Tr}\big(F_0 \wedge \dot{A}_0\big) \tag{6.1.9}$$

$$- \frac{k}{8\pi^2}\frac{d}{dt}\Big|_{t=0}\int_0^t ds \int_{\partial M} \mathrm{Tr}\big(A_s \wedge \dot{A}_s\big) \tag{6.1.10}$$

$$= 2\frac{k}{8\pi^2}\int_M \mathrm{Tr}\big(F_0 \wedge \dot{A}_0\big) - \frac{k}{8\pi^2}\int_{\partial M} \mathrm{Tr}\big(A_0 \wedge \dot{A}_0\big). \tag{6.1.11}$$

which by physicists is usually written

$$\delta S = 2\frac{k}{8\pi^2}\int_M \mathrm{Tr}\big(F \wedge \delta A\big) - \frac{k}{8\pi^2}\int_{\partial M} \mathrm{Tr}\big(A \wedge \delta A\big). \tag{6.1.12}$$

The first term gives the usual equation of motion $F = 0$ and the second term gives a boundary contribution which we require to vanish in order to maintain locality. A close look at (6.1.12) shows that the boundary term is exactly the same as in the abelian theory which means that we will have to make the same requirement: the gauge field A must lie in a Lagrangian subspace of the Lie algebra with respect to the invariant form Tr.

This result immediately raises two new questions. First, which Lagrangian subspaces are there and second, what are the corresponding boundary gauge groups. We will address these questions in the interesting special case of $SU(2) \times SU(2)$ Chern-Simons theory which as we already explained tells us something about surface operators in the $SU(2)$ theory (and thus about $SU(2)$-WZW models).

6.2. Boundary gauge groups

Let us now consider $SU(2)$ Chern-Simons theory on a closed three-manifold M with a surface operator inserted on a surface $\Sigma \subset M$. In $SU(2)$ Chern-Simons theory every ad-invariant bilinear form on the Lie algebra is proportional to the Killing form B since $\mathfrak{su}(2)$ is simple. Folding the theory across Σ , we obtain a $SU(2) \times SU(2)$-theory defined on a manifold with boundary Σ. The action is given in terms of the new bilinear form

$$\mathrm{Tr}_{SU(2)\times SU(2)} \sim B \oplus (-B) \tag{6.2.1}$$

on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

There is one very obvious Lagrangian subspace with respect to this bilinear form given by the diagonal

$$\Delta = \{(u, u) \,|\, u \in \mathfrak{su}(2)\}. \tag{6.2.2}$$

Note that in the non-abelian case the anti-diagonal $\{(u,-u)\,|\,u\in\mathfrak{su}(2)\}$ is *not* a sensible choice for a Lagrangian subspace since it is not a Lie subalgebra of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. Indeed, one has

$$\begin{aligned}\big[(u,-u),(v,-v)\big] &= \big([u,v],[-u,-v]\big) && (6.2.3)\\ &= \big([u,v],[u,v]\big) && (6.2.4)\\ &\neq \big([u,v],-[u,v]\big). && (6.2.5)\end{aligned}$$

But in order to be the Lie algebra of a boundary gauge group, the Lagrangian subspace has to be a Lie subalgebra of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

We will focus on the diagonal subspace Δ and address the question about the corresponding boundary gauge groups. What subgroups are there in $SU(2)\times SU(2)$ whose Lie algebra is Δ?

First we note that the identity component can be nothing else but the diagonal in $SU(2)\times SU(2)$

$$D := \{(U,U)\,|\,U\in SU(2)\}\subset SU(2)\times SU(2) \qquad (6.2.6)$$

which is isomorphic to $SU(2)$. This is due to a standard result in the theory of Lie groups and Lie algebras; see e.g. Theorem 4.14 in [19]. So we only have the freedom to add connected components.

To find further constraints to this freedom, suppose we have already found some disconnected subgroup H with Lie algebra $\mathrm{Lie}(H)=\Delta$. Since the identity component of a Lie group is always a normal subgroup, the group H must actually be a subgroup of the normalizer $N(D)\subset SU(2)\times SU(2)$. This observation generalizes straightforwardly to other gauge groups and in many cases suffices to determine the possible boundary gauge groups completely as we will demonstrate now in the $SU(2)$ case.

For convenience, we will write $SU(2)\times SU(2)$ elements as block matrices. We have

$$\begin{aligned}
&\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} && \in && N(D) \\
\Leftrightarrow\quad &\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} && \in && D \quad \forall U \in SU(2) \\
\Leftrightarrow\quad &\begin{pmatrix} aUa^{-1} & 0 \\ 0 & bUb^{-1} \end{pmatrix} && \in && D \quad \forall U \in SU(2) \\
\Leftrightarrow\quad & aUa^{-1} = bUb^{-1} && && \forall U \in SU(2) \\
\Leftrightarrow\quad & (a^{-1}b)U = U(a^{-1}b) && && \forall U \in SU(2) \\
\Leftrightarrow\quad & a^{-1}b \in Z(SU(2))
\end{aligned} \tag{6.2.7}$$

But the center of $SU(2)$ happens to be very small. In fact, we have

$$Z(SU(2)) = \{\mathbf{1}, -\mathbf{1}\} \cong \mathbb{Z}_2. \tag{6.2.8}$$

This means $a^{-1}b = \mathbf{1}$ or $a^{-1}b = -\mathbf{1}$, or in other words

$$a = \pm b. \tag{6.2.9}$$

This gives us a surprisingly simple characterization of the possible boundary gauge groups. There are only two of them, one being D itself and the other one being

$$N(D) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \pm a \end{pmatrix} \,\middle|\, a \in SU(2) \right\} \cong SU(2) \times \mathbb{Z}_2. \tag{6.2.10}$$

Having solved the $SU(2)$ case, it is straightforward to determine the boundary gauge group for more general gauge groups. We just have to compute the normalizer of the diagonal subgroup D and look for subgroups therein having D as their identity component.

For the slightly more general case $SU(n)\times SU(n)$ for instance, the same calculation as above tells us that the normalizer of the diagonal subgroup is given by matrices $(a,b) \in SU(n)\times SU(n)$ satisfying

$$a^{-1}b \in Z(SU(n)) \cong \mathbb{Z}_n. \tag{6.2.11}$$

which is equivalent to

$$b = e^{2\pi i \frac{k}{n}} a, \qquad k = 0, ..., n-1. \tag{6.2.12}$$

Thus we have

$$N(D) \cong SU(n) \times Z_n \tag{6.2.13}$$

with possible subgroups

$$SU(n) \times \mathbb{Z}_\nu, \qquad \nu | n. \tag{6.2.14}$$

The completely general case with an arbitrary compact gauge group may not in general be solved by merely computing the normalizer of the diagonal subgroup. The problem is that in general $N(D)$ may have a larger dimension than D so that the identity component of $N(D)$ will have a larger Lie algebra than Δ.

A theorem from chapter 20 in [20] gives a method for tackling the general case:

Proposition 6.2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The disconnected Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}$ are precisely the extensions of the connected ones by discrete groups.*

Applied to our case this means, possible boundary gauge groups are groups H fitting into the exact sequence

$$1 \to D \to H \to \Gamma \to 1 \tag{6.2.15}$$

with Γ finite. Yet another way to say this is that one has to find groups $H \subset N(D)$ such that H/D is a finite group. It is easily checked that this is in fact the case for the $SU(n)$ examples above.

6.3. Different Lagrangian subspaces

So far we have only studied one particularly simple Lagrangian subalgebra and the question for other interesting ones suggests itself. In this section we will show that, at least in the $SU(2)$ case, there are essentially no other interesting Lagrangian subalgebras.

We will use the results of [21] of which we will give a short review.

Let G be a connected Lie group and $\mathfrak{g} = \text{Lie } G$ its Lie algebra equipped with a non-degenerate invariant symmetric bilinear form B. Consider the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ equipped with the bilinear form $B \oplus (-B)$.

We call a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ *coisotropic* if $\mathfrak{c}^\perp \subset \mathfrak{c}$ (where $\mathfrak{c}^\perp$ is the orthogonal complement of $\mathfrak{c}$ with respect to B). If a subalgebra $\mathfrak{c}$ is coisotropic, then $\mathfrak{c}^\perp$ is an ideal in $\mathfrak{c}$ and B induces a non-degenerate invariant symmetric bilinear form $\bar{B}$ on $\mathfrak{c}/\mathfrak{c}^\perp$.

The authors of [21] now showed the following (which is called proposition 2.1 in their work):

Proposition 6.3.1. *The set of Lagrangian subalgebras of* $\mathfrak{g} \oplus \mathfrak{g}$ *is in G-equivariant bijection with the set of all triples* $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ *where* $\mathfrak{c}_1$ *and* $\mathfrak{c}_2$ *are coisotropic subalgebras in* $\mathfrak{g}$*, and* $\varphi : \mathfrak{c}_1/\mathfrak{c}_1^\perp \to \mathfrak{c}_2/\mathfrak{c}_2^\perp$ *is an isomorphism preserving the form* $\bar{B}$*.*

Proof. The idea of the proof is the following. Given a Lagrangian subalgebra $\mathfrak{l} \subset \mathfrak{g} \oplus \mathfrak{g}$, put $\mathfrak{c}_1 := \mathrm{pr}_1(\mathfrak{l})$ and $\mathfrak{c}_2 := \mathrm{pr}_2(\mathfrak{l})$ and $\varphi(u + \mathfrak{c}_1^\perp) = v + \mathfrak{c}_2^\perp$ where v is any element such that $(u, v) \in \mathfrak{l}$. Then show that φ has the desired properties. Conversely, given a triple $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ as above, put

$$\mathfrak{l} := \{(u, v) \mid u \in \mathfrak{c}_1,\, v \in \mathfrak{c}_2,\, \varphi(u + \mathfrak{c}_1^\perp) = v + \mathfrak{c}_2^\perp\} \tag{6.3.1}$$

and show that $\mathfrak{l}$ is a Lagrangian subalgebra. □

This proposition gives us a strong constraint on Lagrangian subspaces of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Since $\mathfrak{su}(2)$ is simple, any symmetric invariant bilinear form is a multiple of the Killing form and since the group $SU(2)$ is compact, the Killing form is negative definite. But for a Lie algebra $\mathfrak{g}$ with a definite bilinear form the only coisotropic subalgebra is $\mathfrak{g}$ itself. The proposition, together with (6.3.1) implies that any Lagrangian subalgebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is of the form

$$\mathfrak{l} = \{(u, \varphi(u)) \mid u \in \mathfrak{su}(2)\} \tag{6.3.2}$$

with $\varphi \in \mathrm{Aut}(\mathfrak{su}(2))$.

The diagonal subspace from the previous section corresponds to $\varphi = \mathrm{id}$. Now we can again determine the corresponding boundary gauge groups. Since $SU(2)$ is simply connected, there is a unique group homomorphism $\psi : SU(2) \to SU(2)$ such that $\varphi = d\psi$. Since φ is an isomorphism, ψ is a covering map and using simply connectedness again we see that ψ has to be the universal cover and therefore an isomorphism itself. We can use ψ to do the same calculation as in (6.2.7) to end up with

$$\psi(U)\, \psi(a)^{-1} b = \psi(a)^{-1} b\, \psi(U). \tag{6.3.3}$$

Since ψ is an automorphism, this implies

$$\psi(a)^{-1} b \in Z(SU(2))$$
$$\Rightarrow b = \pm\psi(a)$$

So the normalizer of the identity component is

$$N = \left\{ \begin{pmatrix} a & 0 \\ 0 & \pm\psi(a) \end{pmatrix} \middle| \, a \in SU(2) \right\} \cong SU(2) \times \mathbb{Z}_2 \tag{6.3.4}$$

with subgroups $\{(a, \psi(a)) | \, a \in SU(2)\} \cong SU(2)$ and N itself. This shows that even if we allow for arbitrary Lagrangian subspaces, the boundary gauge groups do not change. Thus, *all* surface operators in $SU(2)$ Chern-simons theory create either $SU(2)$ or $SU(2) \times \mathbb{Z}_2$.[1] This result is important for the next section where we use it to prove a classification result.

[1] But note that this statement was derived in the classical theory. In the quantum theory there might arise surface operators which behave differently.

7. On ADE Classification

7.1. Conformal field theory

As we already mentioned, the $SU(2)$ case is particularly interesting because of the intimate relationship between Chern-Simons theory and conformal field theory.

It is a popular question to ask for a classification of all rational conformal field theories based on a given chiral algebra, i.e. to give a list of all possible Hilbert spaces

$$\mathcal{H} = \bigoplus_{j\bar{j}\in\mathcal{I}} N_{j\bar{j}} \mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}, \tag{7.1.1}$$

where the $\mathcal{V}_j, \mathcal{V}_{\bar{j}}$ are irreducible representations of the chiral algebra and $\mathcal{I}$ is a finite index set. This classification problem is explained in [2] and [15] to which we refer for details.

The structure of the Hilbert space is encoded in the torus partition function

$$Z(\tau) = \sum_{j\bar{j}} N_{j\bar{j}} \chi_j(q) \chi_{\bar{j}}(\bar{q}), \qquad q = e^{2\pi i \tau} \tag{7.1.2}$$

where $\chi_j(q) = \mathrm{Tr}_{\mathcal{V}_j}\left(q^{L_0 - \frac{c}{24}}\right)$ are the characters of the representation and τ is the modular parameter of the torus. For certain chiral algebras (including $\widehat{\mathfrak{su}}(2)_k$), modular invariance of the partition function can be used to determine the possible partition functions completely.

In the case of $\widehat{\mathfrak{su}}(2)_k$, the possible partition functions turn out to be in a one-to-one correspondence with the ADE Dynkin diagrams. In particular, the theories corresponding to the A diagrams exist for all levels $k \geq 0$ and the D invariants can only exist for even k (see table 7.1). Concrete realizations of the A- and D-invariants are given by $SU(2)$ and $SO(3)$ Wess-Zumino-Witten models, respectively. Indeed, it is well-known that $SO(3)$ WZW models only exist for even k [22].

Now let us turn back to the 3D theory. It is known that Chern-Simons theory on a manifold with boundary Σ leads to a conformal field theory on Σ. But since different boundary conditions lead to different CFTs, the ADE classification should carry over to boundary conditions (and therefore also for surface operators) in the 3d theory.

level	diagram	modular invariant Z
$k \geq 0$	A_{k+1}	$\sum_{n=0}^{k} \lvert\chi_n\rvert^2$
$k = 4l$	D_{2l+2}	$\sum_{\substack{n=0 \\ n\in 2\mathbb{Z}}}^{2l-2} \lvert\chi_n + \chi_{4l-n}\rvert^2 + 2\lvert\chi_{2l}\rvert^2$
$k = 4l-2$	D_{2l+1}	$\sum_{\substack{n=0 \\ n\in 2\mathbb{Z}}}^{4l-2} \lvert\chi_n\rvert^2 + \lvert\chi_{2l-1}\rvert^2 + \sum_{\substack{n=1 \\ n\in 2\mathbb{Z}+1}}^{2l-3} (\chi_n\bar{\chi}_{4l-2-n} + \chi_{4l-2-n}\bar{\chi}_n)$
$k = 10$	E_6	$\lvert\chi_0 + \chi_6\rvert^2 + \lvert\chi_3 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_{10}\rvert^2$
$k = 16$	E_7	$\lvert\chi_0 + \chi_{16}\rvert^2 + \lvert\chi_4 + \chi_{12}\rvert^2 + \lvert\chi_6 + \chi_{10}\rvert^2 + \lvert\chi_8\rvert^2$ $+\chi_8(\bar{\chi}_2 + \bar{\chi}_{14}) + (\chi_2 + \chi_{14})\bar{\chi}_8$
$k = 28$	E_8	$\lvert\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}\rvert^2 + \lvert\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}\rvert^2$

Table 7.1.: ADE classification of $\widehat{\mathfrak{su}}(2)_k$ CFTs (taken from [15]).

It is also well-known that $SU(2)$ Chern-Simons theory leads to a $SU(2)$ WZW model on Σ. $SU(2) \times \mathbb{Z}_2$ Chern-Simons theory leads to a $\mathbb{Z}_2$-orbifold of a $SU(2)$-WZW model[1] which is known to be a $SO(3)$-WZW model. The $SU(2)$-WZW model exists for every k and corresponds to the A-type Dynkin diagrams. The $SO(3)$-model exists only for even k and corresponds to type D.

Thus we have found the surface operators in Chern-Simons theory corresponding to the A and D invariants. But if this is true, the $SU(2) \times \mathbb{Z}_2$-surface operator may only exist for even k. It is our next goal to proof this.

7.2. ADE from the three-dimensional Point of View

7.2.1. $SU(2)$ Gauge Group.

In the previous section we have seen that the D-type $\widehat{\mathfrak{su}}(2)_k$ conformal field theories can only exist for even k. In this section, we will make contact to surface operators in Chern-Simons theory. We expect the $SU(2) \times \mathbb{Z}_2$ surface operator to exist only for even k as well and want to show this explicitly.

To do so, recall from section 6.2 that this surface operator allows gauge transformations $U : M \to SU(2)$ to change its sign on Σ. Let us describe this in more detail. Denote by M_1 and M_2 the pieces of M lying on the left, resp. on the right of Σ. Then, the surface

[1] We have already encountered a similar phenomenon. Compare to section 5.4.1.

operator glues the $SU(2)$-bundles over M_1 and M_2 along Σ by the condition that gauge transformations may change their sign on Σ. Locally, we can describe this as follows.

Let $U \subset M_1$ and $V \subset M_2$ be patches over which the bundles are trivial and which have some nonempty intersection $X \cap Y \subset \Sigma$. Elements of the total spaces can be written (x, U) resp. (y, U'), with $x \in X, y \in Y$ and $U, U' \in SU(2)$.

Now, the gluing can be performed by introducing the equivalence relation

$$(x, U) \sim (y, U') \Leftrightarrow x = y \in U \cap V \text{ and } U = \pm U'. \tag{7.2.1}$$

The new bundle is obtained by modding out this relation. Thus, it consists of equivalence classes $[(x, U)] = \{(x, U), (x, -U)\}$. But in this new bundle, all information about the center of the $SU(2)$ gauge group is lost! Only information from the group $SU(2)/\mathbb{Z}_2$ is visible.

In other words, we are left with a $SU(2)/\mathbb{Z}_2 = SO(3)$-bundle.

Now, let us see what implications this has on gauge invariance of the theory. Recall that under a gauge transformation, the Chern-Simons action changes by a Wess-Zumino term (see (A.3.5)). In our $SU(2)$-adapted normalization, this term reads

$$\frac{k}{24\pi^2} \int_M \mathrm{Tr}\left(U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU\right) \tag{7.2.2}$$

(the motivated reader may have a look at [23] for more details on this). It is well-known that in this normalization the $SO(3)$-WZW term can only be an integer (and therefore vanish modulo 1) if k is even [22],[23]. This is related to the fact that the volume of $SO(3)$ is half the volume of $SU(2)$. Indeed, if we recall the facts that $U^{-1}dU = U^*\theta$ (where θ is the Maurer-Cartan form) and that $\theta \wedge \theta \wedge \theta$ is the volume form of $SU(2)$, applying the integral substitution law for manifolds makes it clear that the WZW-term is related to the volume of the group.

But this shows that the $SU(2) \times \mathbb{Z}_2$ surface operator can only be consistent is the level k is an even integer, in accordance with the ADE classification result of the previous section.

Remark: In section 6.3 we showed that there there are more Lagrangian subalgebras than the diagonal in $SU(2) \times SU(2)$. More precisely, we saw that every Lagrangian subalgebra is the Lie algebra of a subgroup of the form

$$\{(U, \pm\psi(U)| \, U \in SU(2)\} \tag{7.2.3}$$

for some automorphism ψ.

So far we only classified the surface operator corresponding to $\psi = \mathrm{id}$. A priori, it might be possible for surface operators with $\psi \neq \mathrm{id}$ to exist also for odd k. However, going back to (7.2.1) one can easily check that inserting ψ into the relation "$\sim$" will not change the fact that we will end up with a $SO(3)$-bundle. Thus, in the case $\psi \neq \mathrm{id}$ the same reasoning as before will show that the corresponding surface operators exist only for even k.

7.2.2. More general gauge groups.

The previous analysis is straightforwardly generalized to higher $SU(n)$ geuge groups. Recall that in these cases, surface operators create boundary gauge groups of the form $SU(n) \times \mathbb{Z}_\nu$ where ν divides n.

Applying the same reasoning as above, we see that these operators lead to a $SU(n)/\mathbb{Z}_\nu$-bundle. Thus, now the WZW-term (7.2.2) with gauge group $SU(n)/\mathbb{Z}_\nu$ is required to vanish modulo 1. This again gives restrictions on the level.

The above procedure demonstrates very directly the relationship between Chern-Simons theory and WZW models. We can see very explicitly that surface operators in Chern-Simons theory exist is and only if the corresponding WZW-action exists.

As a final interesting application, let us apply the above method to surface operators in $SU(3)$ Chern-Simons theory. Since 3 is a prime number, there are (up to trivial twistings by automorphisms) only two surface operators in this theory: the trivial surface operator which does not do anything (and exists for all levels) and a surface operator creating a $SU(3) \times \mathbb{Z}_3$ boundary gauge group.

In this case, we are dealing with a $SU(3)/\mathbb{Z}_3$ WZW-term (normalized with respect to $SU(3)$). It was shown in [22] that this term is an integer for *every* k. Thus, we see that in $SU(3)$ Chern-Simons theory there are no level restrictions on surface operators whatsoever.

To find restrictions on the more complicated surface operators with larger n, it is now clear how to proceed. One has to determine the value of the $SU(n)/\mathbb{Z}_\nu$-WZW-term normalized with respect to $SU(n)$ and see for which k this is an integer. There seem to be good chances that this problem can be tackled using group cohomological methods [23]. We will not pursue this here.

8. Gluing theories with different levels

So far we considered only surface operators gluing two theories which have the same level. A priori there might as well be interesting surface operators between theories with different levels. We will argue that for simple Lie algebras $\mathfrak{g}$ this is not the case. Let us go back to the very beginning when we looked for Lagrangian subspaces of the Lie algebra. We started with a Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ equipped with the invariant bilinear form $B \oplus (-B)$ and characterized Lagrangian subalgebras with respect to this form.

If we glue two theories with different levels, we have to find Lagrangian subspaces with respect to a form

$$B \oplus (-\alpha B) \tag{8.0.1}$$

for some number α. It is immediately clear that there is no more obvious solution such as the diagonal subspace as soon as $\alpha \neq 1$. We will use a reasoning similar to the proof of prop. 2.1 in [21] to show that there are in fact no solutions at all.[1]

Let $\mathfrak{g}$ be a simple Lie algebra and suppose we found a Lagrangian subalgebra $\mathfrak{l} \subset \mathfrak{g} \oplus \mathfrak{g}$ with respect to (8.0.1). Then the projections $\mathfrak{c}_1 = \mathrm{pr}_1(\mathfrak{l})$ and $\mathfrak{c}_2 = \mathrm{pr}_2(\mathfrak{l})$ are coisotropic subalgebras of $\mathfrak{g}$. Now consider the map

$$\varphi : \mathfrak{c}_1/\mathfrak{c}_1^{\perp} \to \mathfrak{c}_2/\mathfrak{c}_2^{\perp} \tag{8.0.2}$$

defined by $\varphi(x + \mathfrak{c}_1^{\perp}) = y + \mathfrak{c}_2^{\perp}$, where $y \in \mathfrak{c}_2$ is any element such that $(x, y) \in \mathfrak{l}$. Then φ is a well-defined isomorphism of Lie algebras. The proof for this is precisely the same as in [21]. It does not, however, preserve the induced bilinear form. It changes it as

$$\bar{B}\big(\varphi(x + \mathfrak{c}_1^{\perp}), \varphi(x' + \mathfrak{c}_1^{\perp})\big) = \frac{1}{\alpha}\bar{B}\big(x + \mathfrak{c}_1^{\perp}, x' + \mathfrak{c}_1^{\perp}\big) \tag{8.0.3}$$

Now, in our case B is definite (and so is αB) so the only coisotropic subalgebra is $\mathfrak{g}$ itself implying that $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{g}$. So the above definition would give us an automorphism of $\mathfrak{g}$ such that

$$B\big(\varphi(x), \varphi(x')\big) = \frac{1}{\alpha}B(x, x') \tag{8.0.4}$$

[1] There are, of course, always *isotropic* subspaces, that is, subspaces on which $B \oplus (-B)$ vanishes. But we are looking for such subspaces of *maximal dimension.*

But this is impossible since B is necessarily a multiple of the Killing form which is invariant under automorphisms of the algebra.

Thus there can be no Lagrangian subspaces with respect to the form $B \oplus (-\alpha B)$ unless $\alpha = 1$.

9. Conclusion

In this last section, we give a brief summation of the our results and pose questions which are still open.

In abelian Chern-Simons theory we investigated the relationship between surface operators and two-dimensional conformal field theory. We computed what their fusion means on the CFT level and confirmed the results of Kapustin and Saulina by showing that for two given $\hat{\mathfrak{u}}(1)$ CFTs with Hilbertspaces $\mathcal{H}^{(p,q)}$ and $\mathcal{H}'^{(p',q')}$ (where (p,q) denotes the choice of decomposition) we get

$$\mathcal{H}^{(p,q)} \circ \mathcal{H}^{(p',q')} = G\,\mathcal{H}^{(p'',q'')}, \tag{9.0.1}$$

where $\circ$ denotes fusion and $G = \gcd(p,q,p',q'),\ q'' = \mathrm{lcm}(\gcd(p,q'),\gcd(q,p'))$.

Moreover, we have explicitly shown the correspondence between primary vertex operators and line operators in $U(1)$ Chern-Simons theory. We have seen that both are determined by a pair of integers and transform the same way under $\mathbb{Z}_v$-transformations generated by the surface operator. We have also seen how the discriminant group which classifies line operators in $U(1)$ Chern-Simons theory shows up in the CFT.

Furthermore, in non-abelian Chern-Simons theory we have investigated which surface operators are possible and for which values of the level k they may appear. In the interesting special case of $SU(2)$ Chern-Simons theory, we have shown that (classically) every surface operator creates either a $SU(2)$ or a $SU(2) \times \mathbb{Z}_2$ gauge group. This fact has been used, together with gauge invariance, to show that the $SU(2) \times \mathbb{Z}_2$ surface operators can only exist for even k, thereby confirming an expectation motivated by the ADE classification of $\hat{\mathfrak{su}}(2)_k$conformal field theories. The same technique was applied to $SU(3)$ Chern-Simons theory to confirm the well-known fact that no level dependence arises.

Finally, we showed that two Chern-Simons theories with different level cannot be glued by a surface operator, at least if their gauge group is a simple Lie group because no Lagrangian subspace with respect to the form $B \oplus (\alpha B)$ can be found unless $\alpha = 1$.

There are several questions which were left unaddressed or raised by this thesis. First of all, ADE classification for $\hat{\mathfrak{su}}(2)_k$ conformal field theories contains not only the A

and D invariants discussed by us, but also E invariants which exist only for the three values $k = 10, 16, 20$. So we would expect there to be additional surface operators in Chern-Simons theory which only exist for $k = 10, 16, 20$. These operators do not seem to be visible in our classical approach and are expected to arise quantum mechanically. So far, these operators have not been found and it would be interesting to understand the three-dimensional reasons why they can only appear at these special values of k.

Secondly, it would be interesting to extend our results to more general gauge groups, for instance those which are not necessarily simple or semisimple. In particular, there might be nontrivial Lagrangian subspaces in $\mathfrak{g} \oplus \mathfrak{g}$ with respect to $B \oplus (\alpha B)$ if g is not simple.

Moreover, to the authors knowledge, the fusion of surface operators in non-abelian Chern-Simons theory is not understood at all so far. As we have seen, there are not many surface operators e.g. in $SU(2)$ Chern-Simons theory so one might hope for a comparatively simple fusion law.

A. Principal bundles, Connections and the Chern-Simons Action

In this section, we give a very short introduction to the theory of principal bundles and connections with the aim to write down a well-defined expression for the Chern-Simons action. This discussion mainly follows [27]

A.1. Basic ingredients

Definition A.1.1. Let G be a Lie group. A triple (P, π, M) of smooth manifolds P, M and a surjective smooth map $\pi : P \to M$ is called a *principal G-bundle*, if

1. G acts on P from the right and one has $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$ and this action is free and transitive on the fibers $\pi^{-1}(x)$ for all $x \in M$.
2. There exists an open covering $\{U_i\}$ of M and diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ such that
   (i) $pr_1 \circ \phi_i = \pi$
   (ii) $\phi_i(p \cdot g) = \phi_i(p) \cdot g$ for all $p \in \pi^{-1}(U_i)$ and $g \in G$.

(the G-action on $U_i \times G$ is defined by $(x, h) \cdot g = (x, hg)$.)

We call M the *base space* and P the *total space* and G the *gauge group* of the bundle.

The simplest example of a principal bundle over a given base M is the *trivial bundle*, given by the triple $(M \times G, pr_1, M)$. Examples of non-trivial bundles with finite gauge groups are given by regular covering spaces.

Definition A.1.2. A *section* of a principal bundle is a smooth map $s : M \to P$ such that $\pi \circ s = \mathrm{id}_M$.

A map which has the same property, but is defined only on an open subset of M is called a *local section.*

We immediately see that sections are always injective. To get a feeling, it is again enlightening to consider the trivial bundle. Here, a section is a map $s = (s_1, s_2) : M \to M \times G$ satisfying $pr_1(s(x)) = x$. So s must have the form $s(x) = (x, s_2(x))$. Thus we have shown that all sections of the trivial bundle are uniquely determined by their second component and therefore are in one-to-one correspondence with maps from M to G.

Definition A.1.3. Let (P, π, M) and (P', π', M') be principal G-bundles. A *morphism* from (P, π, M) to (P', π', M') is a smooth map $\phi : P \to P'$ commuting with the G action. The induced map $\bar{\phi} : M \to M'$ obtained by putting $\bar{\phi}(\pi(p)) = \pi'(\phi(p))$ is said to be *covered* by ϕ.

If $(P, \pi, M) = (P', \pi', M')$ and $\bar{\phi}$ is the identity we call ϕ a *gauge transformation*.

Gauge transformations turn out to take a specific form in general. Since they commute with the G-action, they map fibers into fibers so there exists a map $g_\phi : P \to G$ such that

$$\phi(p) = p \cdot g_\phi(p). \tag{A.1.1}$$

There is a useful connection between gauge transformations and sections of a principal bundle which we like to close this section with.

Lemma A.1.1. *Let (P, π, M) be a principal bundle and s_1, s_2 sections. Then there is a gauge transformation ϕ such that*

$$s_2 = \phi \circ s_1. \tag{A.1.2}$$

Proof. On the image of s_1 we can put $\phi(p) = s_2 \circ s_1^{-1}(p)$, since sections are always injective. Now, every other point in P lies in the fiber over some $m \in M$ and thus has the form $p = s_1(m) \cdot g_p$ for suitable $g_p \in G$ (the mapping $p \mapsto g_p$ is smooth since s_1 and the G-action are smooth). For those points, we put $\phi(p) = s_2(m) \cdot g_p$.

In this way, we have obtained a well-defined smooth map from P to P which manifestly has the required property (A.1.2). We have to check that it is a gauge transformation.

That ϕ commutes with the group action is clear from its construction. It remains to show that the induced map on M is the identity. Let $m \in M$. We compute

$$\bar{\phi}(m) = \bar{\phi}(\pi \circ s_1(m)) = \pi(\phi(s_1(m))) = \pi(s_2(m)) = m$$

which completes the proof. □

A.2. Connections

We let again G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. We denote by L_g and R_g left- and right multiplication with $g \in G$. Recall that every $X \in \mathfrak{g}$ uniquely determines a left invariant vector field $\tilde{X}$ on G by defining

$$\tilde{X}(g) = (L_g)_* X \tag{A.2.1}$$

and that the Maurer-Cartan form θ on a Lie group G is the Lie algebra valued 1-form defined by

$$\theta(X) = (L_{g^{-1}})_* X \tag{A.2.2}$$

for $X \in T_g G$. It satisfies the *Maurer-Cartan equation*

$$d\theta + \frac{1}{2}[\theta, \theta] = 0. \tag{A.2.3}$$

Definition A.2.1. Let $i_p : G \to P$ be the inclusion of fibers, $i_p(g) = g \cdot p$. A *connection* on a principal G-bundle (P, π, M) is a Lie algebra valued 1-form $A \in \Omega^1(P) \otimes \mathfrak{g}$ such that

1. $R_g^* A = \mathrm{Ad}_{g^{-1}} \circ A \qquad \forall g \in G$
2. $i_p^* A = \theta \qquad \forall p \in P$

It is important to know the behavior of a connection under pullbacks with gauge transformations. We shall investigate this now. In the following we will often denote principal bundles with P instead of (P, π, M) when no ambiguities can arise.

Proposition A.2.1. *Let A be a connection on a principal bundle P and $\phi : P \to P$ a gauge transformation. Then one has*

$$\phi^* A = \mathrm{Ad}_{g_\phi^{-1}} \circ A + g_\phi^* \theta, \tag{A.2.4}$$

where θ is the Maurer-Cartan form.

Proof. Writing out the pullback we get

$$\phi^* A(X) = A(\phi_* X)$$

for some $X \in T_p P$. So we have to calculate the push forward ϕ_*. Let γ be some curve in P such that $\dot{\gamma}(0) = X$. Then

$$\begin{aligned} \phi_*(X) &= \left.\frac{d}{dt}\right|_{t=0} \phi(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \gamma(t) \cdot g_\phi(\gamma(t)) \\ &= (R_{g_\phi(p)})_*(X) + \widetilde{(L_{g_\phi^{-1}})_*(X)}(p \cdot g), \end{aligned}$$

where in the third line we used a standard identity whose proof is elementary. Applying A, we find

$$\phi^* A(X) = (R_{g_\phi(p)})^* A(X) + g_\phi^* \theta(X).$$

Now use 1. of definition A.2.1. □

If G is a matrix Lie group one has $\mathrm{Ad}_g \circ A = gAg^{-1}$ (matrix product) and $\theta = g^{-1}dg$. In this case, the transformation law (A.2.4) turns into

$$\phi^* A = g^{-1}Ag + g^{-1}dg \tag{A.2.5}$$

which physicists tend to like more.

Finally, we introduce the *curvature* of a connection A defined as

$$F(A) = dA + \frac{1}{2}[A, A]. \tag{A.2.6}$$

It can be checked to transform as

$$\phi^* F = \mathrm{Ad}_{g_\phi^{-1}} \circ F$$

under gauge transformations. In particular, the trace of F is gauge invariant.

A.3. The Chern-Simons Action

Now we are ready to construct the Chern-Simons action. Let P be a principal bundle over a closed 3-manifold M and let A be a connection on that bundle. Choose an ad-invariant bilinear form

$$\langle \cdot, \cdot \rangle : \mathfrak{g} \oplus \mathfrak{g} \to \mathbb{R}$$

on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. We define the *Chern-Simons form* to be

$$\alpha(A) = \langle A, F(A) \rangle - \frac{1}{6}\langle A, [A, A] \rangle. \tag{A.3.1}$$

This is a 3-form and so can be integrated over a 3-manifold. Indeed, the Chern-Simons action will be an integral over this form, but before we write it down we need to know how the Chern-Simons form behaves under gauge transformations.

Lemma A.3.1. *Under gauge transformations the Chern-Simons form transforms as*

$$\phi^* \alpha = \alpha + d\big\langle \mathrm{Ad}_{g_\phi^{-1}} A \, , \, g_\phi^* \theta \big\rangle - \frac{1}{6}\big\langle g_\phi^* \theta \, , \, [g_\phi^* \theta, g_\phi^* \theta] \big\rangle \tag{A.3.2}$$

Proof. This is a rather straightforward calculation repeatedly making use of the Maurer-cartan equation and the ad-invariance of $\langle \cdot\,,\, \cdot \rangle$. First, we have

$$\begin{aligned}
\phi^*\alpha &= \big\langle \mathrm{Ad}_{g_\phi^{-1}} A + g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} F \big\rangle - \frac{1}{6}\big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, [\mathrm{Ad}_{g_\phi^{-1}} A + g_\phi^*\theta, \mathrm{Ad}_{g_\phi^{-1}} A + g_\phi^*\theta] \big\rangle \\
&= \big\langle A\,,\, F \big\rangle - \frac{1}{6}\big\langle g_\phi^*\theta\,,\, [g_\phi^*\theta, g_\phi^*\theta] \big\rangle - \frac{1}{6}\big\langle A\,,\, [A, A] \big\rangle + \big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} F \big\rangle \\
&\quad - \frac{1}{3}\big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, [g_\phi^*\theta, \mathrm{Ad}_{g_\phi^{-1}} A] \big\rangle - \frac{1}{6}\big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, [g_\phi^*\theta, g_\phi^*\theta] \big\rangle \\
&\quad - \frac{1}{3}\big\langle g_\phi^*\theta\,,\, [g_\phi^*\theta, \mathrm{Ad}_{g_\phi^{-1}} A] \big\rangle - \frac{1}{6}\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle
\end{aligned}$$

The first three terms are already part of what we want. The last two lines in the above can be brought into the form

$$-\frac{1}{2}\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle + \big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, d(g_\phi^*\theta) \big\rangle \tag{A.3.3}$$

using ad-invariance and the Maurer-Cartan equation. Thus we have

$$\begin{aligned}
\phi^*\alpha = \alpha &- \frac{1}{6}\big\langle g_\phi^*\theta\,,\, [g_\phi^*\theta, g_\phi^*\theta] \big\rangle - \frac{1}{2}\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle + \big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, d(g_\phi^*\theta) \big\rangle \\
&+ \big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} F \big\rangle.
\end{aligned}$$

Let us now compute the term $\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} F \big\rangle$. We have

$$\begin{aligned}
\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} F \big\rangle &= \big\langle g_\phi^*\theta\,,\, d(\mathrm{Ad}_{g_\phi^{-1}} A) + \frac{1}{2}\mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle \\
&= \big\langle g_\phi^*\theta\,,\, d(\mathrm{Ad}_{g_\phi^{-1}} A) \big\rangle + \frac{1}{2}\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle \\
&= d\big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, g_\phi^*\theta \big\rangle - \big\langle \mathrm{Ad}_{g_\phi^{-1}} A\,,\, d(g_\phi^*\theta) \big\rangle + \frac{1}{2}\big\langle g_\phi^*\theta\,,\, \mathrm{Ad}_{g_\phi^{-1}} [A, A] \big\rangle
\end{aligned}$$

The last two terms exactly cancel those in (A.3.3) leaving us precisely with (A.3.2). □

Now we are ready to write down the action.

Definition A.3.1. Let P be a principal G-bundle and s a section. The *Chern-Simons action* is defined as

$$S(s, A) = \int_M s^*\alpha(A). \tag{A.3.4}$$

Several remarks are in order. First, we would like the action to be independent of s, of course, which is just gauge invariance. Second, one should note that it is not clear if there even *exists* a section for a given principal bundle P. In fact, one can show that sections exist if and only if the bundle is trivial. Therefore, for the remainder of this section we will assume that P is trivial.[1] The case where P may be nontrivial is treated in section [23].

Now let us turn to gauge invariance.

Proposition A.3.1. *Let s, s' be sections of P and ϕ a gauge transformation such that $s' = \phi \circ s$. Then the following equation holds*

$$S(s', A) = S(s, A) - \int_M \frac{1}{6}\, g_\phi^* \langle \theta, [\theta, \theta] \rangle. \tag{A.3.5}$$

Proof. Since M is closed, this follows immediately from Lemma A.3.1. □

Now, this was a very confident way of saying that S is *not* gauge invariant. What can we do about this? The key point is to realize that to write down a gauge invariant path integral, the action itself does not have to be gauge invariant but only its exponential $e^{2\pi i S}$. In other words, S only has to be gauge invariant *modulo one*. This motivates the following

Hypothesis 1. Assume the bilinear form $\langle \cdot, \cdot \rangle$ to be normalized such that the closed form $-\frac{1}{6}\langle \theta, [\theta, \theta] \rangle$ represents an integral cohomology class.

Now we can define the final Chern-Simons action as

$$S(A) := S(s, A) \qquad (\text{mod } 1). \tag{A.3.6}$$

In the special case of $SU(2)$, every ad-invariant bilinear form in the Lie algebra is a multiple of the Killing form and the ones that are integral have the form

$$\langle a, b \rangle = -\frac{k}{8\pi^2} \mathrm{Tr}(ab), \qquad a, b \in \mathfrak{su}(2), \tag{A.3.7}$$

where k is an integer. In this case, the Chern-Simons action takes the form

$$S(A) = -\frac{k}{8\pi^2} \int_M \mathrm{Tr}\big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\big). \tag{A.3.8}$$

Note that in this section we have only considered the Chern-Simons action on *closed* manifolds. It can be rigorously defined on manifolds with boundary as well, but then further problems with gauge invariance arise, since the term $\int d\langle \mathrm{Ad}_{g_\phi^{-1}} A, g_\phi^* \theta \rangle$ does not vanish anymore. In this case, the upshot is that the action will not be a simple number anymore, but a section in a specific line bundle. The interested reader may consult [27].

[1] For example, this is always true, if the gauge group is simply connected.

B. Non-rational fusion product?

The discussion in section 5.6.2 gives us a pairing on the space of rational $\hat{\mathfrak{u}}(1)$ CFTs (henceforth referred to as the *fusion product*). However, this product depends on the way we decompose the Hilbert spaces into extended modules (see section 5.4). This is somewhat unsatisfactory since we would like such a product to depend only on the CFT and not on the way we write it down. One way to obtain an independent product would be to always take the *minimal* decomposition (5.4.1) of $\mathcal{H}$. This turns out to be problematic because the redundancy in $nq - mp$ now leads to a sum over $\mathbb{Z}$ (instead of $\mathbb{Z}_g$) . Thus, in the coefficient matrix N''_{ab} we would be summing up infinitely many 1's which makes it impossible to interpret the result as a free Boson CFT again.

For non-rational CFTs, however, the charges Q_{mn} determine the numbers m and n uniquely and the above redundancy problem does not emerge. Thus, non-rational CFTs seem to be a better starting point for the definition of an invariant product. We will see that not even in this case, one obtains a sensible product structure. But let us see how far we get.

$\mathbb{Z}^2$-freedom. Let

$$\mathcal{H} = \bigoplus_{m,n} \mathcal{V}_{Q_{mn}} \otimes \overline{\mathcal{V}}_{\bar{Q}_{mn}} \qquad \text{and} \qquad \mathcal{H}' = \bigoplus_{m',n'} \mathcal{V}'_{Q'_{m'n'}} \otimes \overline{\mathcal{V}}'_{\bar{Q}'_{m'n'}} \tag{B.0.1}$$

be two Hilbert spaces of two non-rational CFTs. The fusion product will be a sum over all $Q_{mn}, \bar{Q}'_{m'n'}$ satisfying $Q'_{m'n'} = \bar{Q}_{mn}$. Since the fused CFT sould again have a charge lattice of rank 2, the equation $Q'_{m'n'} = \bar{Q}_{mn}$ must have a $\mathbb{Z}^2$-freedom in m and n. In the following, we will derive a sufficient condition for such a $\mathbb{Z}^2$-freedom.

The equation $Q'_{m'n'} = \bar{Q}_{mn}$ is equivalent to

$$\frac{n'}{R'} - m'R' = \frac{n}{R} + mR. \tag{B.0.2}$$

Putting $a = \frac{R'}{R}$ and $b = RR'$, we can write this as

$$n' - m'ab - mb - na = 0. \tag{B.0.3}$$

Lemma B.0.2. *Suppose that either a^2 or b^2 is rational and there exist numbers $k, n, m \in \mathbb{Z}\backslash\{0\}$ such that*

$$am + bn + k = 0. \tag{B.0.4}$$

Then one has $\mathbb{Z}^2$-freedom in equation (B.0.3).

Proof. Let wlog. $a^2 \in \mathbb{Q}$. Multiply (B.0.4) by a to obtain

$$a^2 m + abn + ak = 0. \tag{B.0.5}$$

Now, write $a^2 = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and multiply by q

$$pm + abnq + akq = 0. \tag{B.0.6}$$

Now, we have found two quadruples of solutions to (B.0.3), namely

$$(k, 0, m, n) \qquad \text{and} \qquad (pm, qn, 0, kq). \tag{B.0.7}$$

These are clearly linearly independent since $n \neq 0$. □

The condition $am + bn + k = 0$ seems to be quite strong, but one can actually show that it is *necessary* for a $\mathbb{Z}^2$-freedom in (B.0.3). It is also easy to see that we can not get more than a $\mathbb{Z}^2$-freedom:

Suppose, we had three linearly independent quadruples of solutions to (B.0.3). By linear combinations, we could form new sets of solutions of the form $(k, 0, 0, m)$ and $(k, 0, n, 0)$. But this means that a and b satisfy

$$am + k = 0 \qquad \text{and} \qquad bn + k = 0 \tag{B.0.8}$$

which immediately implies $a, b \in \mathbb{Q}$ and therefore $R^2, R'^2 \in \mathbb{Q}$.

Coefficient matrix. Now that we have a large class of theories admitting the required $\mathbb{Z}^2$-freedom in (B.0.3), it is straightforward to write down the fused coefficient matrix. Let (n_1, m_1, n_1', m_1') and (n_2, m_2, n_2', m_2') be two linearly independent sets of solutions to $\bar{Q}_{mn} = Q'_{m'n'}$. Put

$$\sqrt{2}Q_{vw} = \frac{vn_1 + wn_2}{R} - (vm_1 + wm_2)R, \qquad \text{and}$$
$$\sqrt{2}Q'_{vw} = \frac{vn_1' + wn_2'}{R'} - (vm_1' + wm_2')R'.$$

Then, the set $\{(Q_{vw}, \bar{Q}'_{vw}) \,|\, v, w \in \mathbb{Z}\}$ is the charge lattice of the fused theory. Its Hilbert space is, of course,

$$\mathcal{H}'' = \bigoplus_{v,w} \mathcal{V}_{Q_{vw}} \otimes \overline{\mathcal{V}}_{\bar{Q}'_{vw}}. \tag{B.0.9}$$

In contrast to the rational case, we have no information about the radius of the fused theory yet. In fact, we do not even know if there exists a radius R'' such that the above charge lattice can be written in the form $\left\{\frac{1}{\sqrt{2}}\left(\frac{v''}{R''} - w''R'', \frac{v''}{R''} + w''R''\right)\middle| v'', w'' \in \mathbb{Z}\right\}$.

We will actually show in the following that such a radius does *not* exist and therefore the above attempt to define an invariant product structure is doomed to fail.

The radius R''. First, let us show again that for $R^2, R'^2 \notin \mathbb{Q}$ the charges Q_{vw} determine the integers v and w uniquely, so we do not have to bother with redundancy. To see this, note that $x := vn_1 + wn_2$ and $y := vm_1 + wm_2$ are uniquely determined by Q_{vw} (otherwise, R^2 would have to be rational). So, if we are able to express v and w in terms of x and y we are done. Using the defining equations of x and y, we obtain

$$\begin{aligned} xm_2 - yn_2 &= v(n_1m_2 - n_2m_1) \\ xm_1 - yn_1 &= w(n_2m_1 - m_2n_1) \end{aligned}$$

so v and w are determined by Q_{vw} if $n_1m_2 - n_2m_1 \neq 0$.

Lemma B.0.3. *If $R'^2 \neq \mathbb{Q}$, then $n_1m_2 - n_2m_1 \neq 0$.*

Proof. Suppose, $n_1m_2 - m_1n_2 = 0$. We can interpret this equation as

$$\det \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = 0 \tag{B.0.10}$$

and deduce that the vectors (n_1, m_1) and (n_2, m_2) are linearly dependent. Thus, there exists a $\lambda \in \mathbb{Q}$ such that $(n_2, m_2) = (\lambda n_1, \lambda m_1)$. Using (B.0.2), we now compute

$$\begin{aligned} \frac{n_2'}{R'} - m_2'R' &= \frac{n_2}{R} + m_2R \\ &= \lambda\left(\frac{n_1}{R} + m_1R\right) \\ &= \lambda\left(\frac{n_1'}{R'} - m_1'R'\right) \end{aligned}$$

We infer

$$\lambda n_1' - n_2' = (\lambda m_1' - m_2')R'^2. \tag{B.0.11}$$

Hence, if $R'^2 \notin \mathbb{Q}$, we must have $(n_2', m_2') = \lambda(n_1', m_1')$ and therefore

$$(n_1, m_1, n_1', m_1') = \lambda(n_2, m_2, n_2', m_2') \tag{B.0.12}$$

contradicting linear independence of (n_1, m_1, n_1', m_1') and (n_2, m_2, n_2', m_2'). □

To find the radius for the fused CFT, we now have to solve the system of equations

$$\frac{vn_1 + wn_2}{R} - (vm_1 + wm_2)R = \frac{v''}{R''} - w''R'' \tag{B.0.13}$$

$$\frac{vn_1' + wn_2'}{R'} + (vm_1' + wm_2')R' = \frac{v''}{R''} + w''R'' \tag{B.0.14}$$

We will now show that this system has no solution. Subtracting eq. (B.0.13) from eq. (B.0.14) and using eq. (B.0.2), we get

$$w''R'' = (vm_1 + wm_2)R + (vm_1' + wm_2')R' \tag{B.0.15}$$

$$\Rightarrow \quad w'' = v\,\frac{m_1R + m_1'R'}{R''} + w\,\frac{m_2R + m_2'R'}{R''} \tag{B.0.16}$$

Since w'' has to be an integer for all v, w, the terms $\frac{m_1R+m_1'R'}{R''}$ and $\frac{m_2R+m_2'R'}{R''}$ have to be integers themselves. In particular

$$\frac{m_1R + m_1'R'}{m_2R + m_2'R'} =: r \in \mathbb{Q} \tag{B.0.17}$$

$$\Leftrightarrow \quad m_1 - rm_2 = (rm_2' - m_1')\frac{R'}{R} \tag{B.0.18}$$

$$\Rightarrow \quad \text{Either } \frac{R'}{R} \in \mathbb{Q}, \quad \text{or } \; m_1m_2' = m_1'm_2 \tag{B.0.19}$$

But as in Lemma B.0.3, one shows that $m_1m_2' = m_1'm_2$ implies $\frac{R'}{R} \in \mathbb{Q}$, so in any case we must have $\frac{R'}{R} \in \mathbb{Q}$.

By analogous arguments we can derive a formula for v'' similar to (B.0.16) and find that in any case the product RR' must be rational. But this contradicts our assumptions since now we must have

$$RR' \cdot \frac{R}{R'} = R^2 \in \mathbb{Q} \tag{B.0.20}$$

$$RR' \cdot \frac{R'}{R} = R'^2 \in \mathbb{Q}. \tag{B.0.21}$$

Hence, both theories would have to be rational which contradicts our initial assumption.

Bibliography

[1] A. Kapustin, N. Saulina, Topological boundary conditions in abelian Chern-Simons theory, Nucl.Phys. B845 (2011) 393–435. arXiv:1008.0654, doi:10.1016/j.nuclphysb.2010.12.017.

[2] A. Cappelli, J.-B. Zuber, A-D-E Classification of Conformal Field TheoriesarXiv:0911.3242.

[3] E. Witten, Quantum Field Theory and the Jones Polynomial, Commun.Math.Phys. 121 (1989) 351. doi:10.1007/BF01217730.

[4] S. Elitzur, G. W. Moore, A. Schwimmer, N. Seiberg, Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory, Nucl.Phys. B326 (1989) 108. doi:10.1016/0550-3213(89)90436-7.

[5] G. W. Moore, N. Seiberg, Taming the Conformal Zoo, Phys.Lett. B220 (1989) 422. doi:10.1016/0370-2693(89)90897-6.

[6] A. Kapustin, N. Saulina, Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field TheoryarXiv:1012.0911.

[7] J. Fuchs, C. Schweigert, A. Valentino, Bicategories for boundary conditions and for surface defects in 3-d TFT, Commun.Math.Phys. 321 (2013) 543–575. arXiv:1203.4568, doi:10.1007/s00220-013-1723-0.

[8] S. D. Stirling, Abelian Chern-Simons theory with toral gauge group, modular tensor categories, and group categoriesarXiv:0807.2857.

[9] K. Gawedzki, Boundary WZW, G/H, G/G and CS theories, Annales Henri Poincare 3 (2002) 847–881. arXiv:hep-th/0108044, doi:10.1007/s00023-002-8639-0.

[10] K. Gawedzki, Conformal field theory: A Case studyarXiv:hep-th/9904145.

[11] A. Kapustin, Topological Field Theory, Higher Categories, and Their ApplicationsarXiv:1004.2307.

[12] A. Kapustin, E. Witten, Electric-Magnetic Duality And The Geometric Langlands Program, Commun.Num.Theor.Phys. 1 (2007) 1–236. arXiv:hep-th/0604151.

[13] S. Gukov, Gauge theory and knot homologies, Fortsch.Phys. 55 (2007) 473–490. arXiv:0706.2369, doi:10.1002/prop.200610385.

[14] A. Kapustin, K. Setter, K. Vyas, Surface Operators in Four-Dimensional Topological Gauge Theory and Langlands DualityarXiv:1002.0385.

[15] P. Francesco, P. Mathieu, D. Senechal, Conformal Field Theory (Graduate Texts in Contemporary Physics), corrected Edition, Springer, 1999.
URL http://amazon.com/o/ASIN/038794785X/

[16] R. Blumenhagen, E. Plauschinn, Introduction to Conformal Field Theory: With Applications to String Theory (Lecture Notes in Physics), 2009th Edition, Springer, 2009.

[17] A. Cappelli, C. Itzykson, J. Zuber, Modular Invariant Partition Functions in Two-Dimensions, Nucl.Phys. B280 (1987) 445–465. doi:10.1016/0550-3213(87)90155-6.

[18] J. Fuchs, I. Runkel, C. Schweigert, TFT construction of RCFT correlators 1. Partition functions, Nucl.Phys. B646 (2002) 353–497. arXiv:hep-th/0204148, doi:10.1016/S0550-3213(02)00744-7.

[19] M. R. Sepanski, Compact Lie Groups (Graduate Texts in Mathematics), 2007th Edition, Springer, 2006.
URL http://amazon.com/o/ASIN/0387302638/

[20] J. M. Lee, Introduction to Smooth Manifolds (Graduate Texts in Mathematics), 1st Edition, Springer, 2002.
URL http://amazon.com/o/ASIN/0387954481/

[21] E. Karolinsky, S. Lyapina, Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ where $\mathfrak{g}$ is a real simple Lie algebra of real rank one, Travaux mathematiques 16 (2005) 229.

[22] D. Gepner, E. Witten, String Theory on Group Manifolds, Nucl.Phys. B278 (1986) 493. doi:10.1016/0550-3213(86)90051-9.

[23] R. Dijkgraaf, E. Witten, Topological Gauge Theories and Group Cohomology, Commun.Math.Phys. 129 (1990) 393. doi:10.1007/BF02096988.

[24] M. Mimura, H. Toda, M. Mimura, H. Toda, Topology of Lie Groups, I and II (Translations of Mathematical Monographs, Vol 91), American Mathematical Society, 2000.
URL http://amazon.com/o/ASIN/0821813420/

[25] D. S. Freed, K. K. Uhlenbeck, Instantons and Four-Manifolds (Mathematical Sciences Research Institute Publications), softcover reprint of the original 1st ed. 1984 Edition, Springer, 2012.
URL http://amazon.com/o/ASIN/1468402609/

[26] W. G. Dwyer, H.-W. Henn, Homotopy Theoretic Methods in Group Cohomology (Advanced Courses in Mathematics - CRM Barcelona), 2001st Edition, Birkhäuser, 2001.
URL http://amazon.com/o/ASIN/3764366052/

[27] D. S. Freed, Classical Chern-Simons theory. Part 1, Adv.Math. 113 (1995) 237–303. arXiv:hep-th/9206021, doi:10.1006/aima.1995.1039.

[28] V. V. Prasolov, Elements of Homology Theory (Graduate Studies in Mathematics), American Mathematical Society, 2007.
URL http://amazon.com/o/ASIN/0821838121/

[29] G. E. Bredon, Topology and Geometry (Graduate Texts in Mathematics), corrected Edition, Springer, 1997.
URL http://amazon.com/o/ASIN/0387979263/

[30] M. F. Atiyah, Topological quantum field theory, Publications Mathématiques de l'IHÉS 68 (1988) 175–186.
URL http://eudml.org/doc/104037

[31] F. Hirzebruch, A. Borel, Characteristic classes and homogeneous spaces. i, American Journal of Mathematics 80 (1958) 458–538.
URL http://hirzebruch.mpim-bonn.mpg.de/162/

[32] H. Baum, Eichfeldtheorie: Eine Einfuhrung in die Differentialgeometrie auf Faserbundeln (Springer-Lehrbuch Masterclass) (German Edition), 2009th Edition, Springer, 2009.
URL http://amazon.com/o/ASIN/3540382925/

[33] K. S. Brown, Cohomology of Groups (Graduate Texts in Mathematics, No. 87), first edition Edition, Springer, 1982.
URL http://amazon.com/o/ASIN/0387906886/

[34] V. Knizhnik, A. Zamolodchikov, Current Algebra and Wess-Zumino Model in Two-Dimensions, Nucl.Phys. B247 (1984) 83–103. doi:10.1016/0550-3213(84)90374-2.

[35] P. Goddard, D. I. Olive, Kac-Moody and Virasoro Algebras in Relation to Quantum Physics, Int.J.Mod.Phys. A1 (1986) 303. doi:10.1142/S0217751X86000149.

[36] D. S. Freed, Remarks on Chern-Simons Theory, ArXiv e-printsarXiv:0808.2507.

[37] D. S. Freed, F. Quinn, Chern-Simons theory with finite gauge group, Commun.Math.Phys. 156 (1993) 435–472. arXiv:hep-th/9111004, doi:10.1007/BF02096860.

[38] J. Fuchs, Fusion rules in conformal field theory, Fortsch.Phys. 42 (1994) 1–48. arXiv:hep-th/9306162.

[39] M. R. Gaberdiel, An Introduction to conformal field theory, Rept.Prog.Phys. 63 (2000) 607–667. arXiv:hep-th/9910156, doi:10.1088/0034-4885/63/4/203.

[40] S. E. Axelrod, Geometric quantization of Chern-Simons gauge theory.

[41] P. H. Ginsparg, Applied Conformal Field TheoryarXiv:hep-th/9108028.

[42] D. Husemöller, M. Joachim, B. Jurco, M. Schottenloher, Basic Bundle Theory and K-Cohomology Invariants (Lecture Notes in Physics), 2008th Edition, Springer, 2007.
URL http://amazon.com/o/ASIN/3540749551/

[43] D. Tong, TASI lectures on solitons: Instantons, monopoles, vortices and kinksarXiv:hep-th/0509216.

[44] W. Lickorish, An Introduction to Knot Theory (Graduate Texts in Mathematics), 1997th Edition, Springer, 1997.
URL http://amazon.com/o/ASIN/038798254X/

[45] M. Manoliu, Abelian Chern-Simons theory, in: eprint arXiv:dg-ga/9610001, 1996, p. 10001.

[46] M. Marino, Chern-Simons theory and topological strings, Rev.Mod.Phys. 77 (2005) 675–720. arXiv:hep-th/0406005, doi:10.1103/RevModPhys.77.675.

[47] J. Milnor, Spin structures on manifolds., Collected Papers of John Milnor: Differential topology 3 (2007) 293.

[48] J. Milnor, J. D. Stasheff, Characteristic Classes. (AM-76), first edition Edition, Princeton University Press, 1974.
URL http://amazon.com/o/ASIN/0691081220/

[49] G. W. Moore, N. Seiberg, Classical and Quantum Conformal Field Theory, Commun.Math.Phys. 123 (1989) 177. doi:10.1007/BF01238857.

[50] M. Schottenloher, A Mathematical Introduction to Conformal Field Theory (Lecture Notes in Physics), 2nd Edition, Springer, 2008.
URL http://amazon.com/o/ASIN/3540686258/

[51] G. Segal, The definition of conformal field theory (2002) 421–575.

[52] A. Borel, Topology of Lie groups and characteristic classes, Bull. Amer. Math. Soc. 61 (1955) 397–432.

[53] E. Witten, Topological quantum field theory, Communications in Mathematical Physics 117 (3) (1988) 353–386.

[54] V. Varadarajan, Lie Groups, Lie Algebras, and Their Representation (Graduate Texts in Mathematics, Vol. 102), 1st Edition, Springer, 1984.
URL http://amazon.com/o/ASIN/0387909699/

Acknowledgements

I would like to thank my supervisor Daniel Roggenkamp for great guidance, helpful discussions and support during my thesis. Furthermore, I would like to thank Eike Fokken and Ingo Roth for many helpful and enlightening discussions some of which were related to physics.

Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, ...

Frank Rösler